Journal of Dynamics and Differential Equations, Vol. 20, No. 2, June 2008 (© 2007)
DOI: 10.1007/s10884-007-9100-3

The Asymptotic Finite-dimensional Character of a Spectrally-hyperviscous Model of 3D Turbulent Flow

Joel Avrin[1]

Received January 2, 2007; revised July 28, 2007

We obtain attractor and inertial-manifold results for a class of 3D turbulent flow models on a periodic spatial domain in which hyperviscous terms are added spectrally to the standard incompressible Navier–Stokes equations (NSE). Let P_m be the projection onto the first m eigenspaces of $A = -\Delta$, let μ and α be positive constants with $\alpha \geqslant 3/2$, and let $Q_m = I - P_m$, then we add to the NSE operators μA_φ in a general family such that $A_\varphi \geqslant Q_m A^\alpha$ in the sense of quadratic forms. The models are motivated by characteristics of spectral eddy-viscosity (SEV) and spectral vanishing viscosity (SVV) models. A distinguished class of our models adds extra hyperviscosity terms only to high wavenumbers past a cutoff λ_{m_0} where $m_0 \leqslant m$, so that for large enough m_0 the inertial-range wavenumbers see only standard NSE viscosity.
We first obtain estimates on the Hausdorff and fractal dimensions of the attractor $\mathcal{A}$ (respectively $\dim_{\mathrm{H}} \mathcal{A}$ and $\dim_{\mathrm{F}} \mathcal{A}$). For a constant K_α on the order of unity we show if $\mu \geqslant \nu$ that $\dim_{\mathrm{H}} \mathcal{A} \leqslant \dim_{\mathrm{F}} \mathcal{A} \leqslant K_\alpha [\lambda_m/\lambda_1]^{9(\alpha-1)/(10\alpha)} [l_0/l_\epsilon]^{(6\alpha+9)/(5\alpha)}$ and if $\mu \leqslant \nu$ that $\dim_{\mathrm{H}} \mathcal{A} \leqslant \dim_{\mathrm{F}} \mathcal{A} \leqslant K_\alpha (\nu/\mu)^{9/(10\alpha)} [\lambda_m/\lambda_1]^{9(\alpha-1)/(10\alpha)} [l_0/l_\epsilon]^{(6\alpha+9)/(5\alpha)}$ where ν is the standard viscosity coefficient, $l_0 = \lambda_1^{-1/2}$ represents characteristic macroscopic length, and l_ϵ is the Kolmogorov length scale, i.e. $l_\epsilon = (\nu^3/\epsilon)$ where ϵ is Kolmogorov's mean rate of dissipation of energy in turbulent flow. All bracketed constants and K_α are dimensionless and scale-invariant. The estimate grows in m due to the term λ_m/λ_1 but at a rate lower than $m^{3/5}$, and the estimate grows in μ as the relative size of ν to μ. The exponent on l_0/l_ϵ is significantly less than the Landau–Lifschitz predicted value of 3. If we impose the condition $\lambda_m \leqslant (1/l_\epsilon)^2$, the estimates become $K_\alpha [l_0/l_\epsilon]^3$ for $\mu \geqslant \nu$ and $K_\alpha (\nu/\mu)^{\frac{9}{10\alpha}} [l_0/l_\epsilon]^3$ for $\mu \leqslant \nu$. This result holds independently of α, with K_α and c_α independent of m. In an SVV example $\mu \geqslant \nu$, and for $\mu \leqslant \nu$ aspects

[1] Department of Mathematics and Statistics, University of North Carolina at Charlotte, Charlotte, USA. E-mail: jdavrin@uncc.edu

of SEV theory and observation suggest setting $\mu \sim c\nu$ for $1/c$ within α orders of magnitude of unity, giving the estimate $c_\alpha K_\alpha [l_0/l_\epsilon]^3$ where c_α is within an order of magnitude of unity. These choices give straight-up or nearly straight-up agreement with the Landau–Lifschitz predictions for the number of degrees of freedom in 3D turbulent flow with m so large that (e.g. in the distinguished-class case for m_0 large enough) we would expect our solutions to be very good if not virtually indistinguishable approximants to standard NSE solutions. We would expect lower choices of λ_m (e.g. $\lambda_m \sim a(1/l_\epsilon)$ with $a > 1$) to still give good NSE approximation with lower powers on l_0/l_ϵ, showing the potential of the model to reduce the number of degrees of freedom needed in practical simulations. For the choice $\epsilon \sim \nu^\alpha$, motivated by the Chapman–Enskog expansion in the case $m = 0$, the condition becomes $\lambda_m \leqslant \nu(1/l_\epsilon)^2$, giving agreement with Landau–Lifschitz for smaller values of λ_m then as above but still large enough to suggest good NSE approximation. Our final results establish the existence of a inertial manifold $\mathcal{M}$ for reasonably wide classes of the above models using the Foias/Sell/Temam theory. The first of these results obtains such an $\mathcal{M}$ of dimension $N > m$ for the general class of operators A_φ if $\alpha > 5/2$.
The special class of A_φ such that $P_m A_\varphi = 0$ and $Q_m A_\varphi \geqslant Q_m A^\alpha$ has a unique spectral-gap property which we can use whenever $\alpha \geqslant 3/2$ to show that we have an inertial manifold $\mathcal{M}$ of dimension m if m is large enough. As a corollary, for most of the cases of the operators A_φ in the distinguished-class case that we expect will be typically used in practice we also obtain an $\mathcal{M}$, now of dimension m_0 for m_0 large enough, though under conditions requiring generally larger m_0 than the m in the special class. In both cases, for large enough m (respectively m_0), we have an inertial manifold for a system in which the inertial range essentially behaves according to standard NSE physics, and in particular trajectories on $\mathcal{M}$ are controlled by essentially NSE dynamics.

KEY WORDS: 3D turbulent flow models; attractor dimension; inertial manifolds; degrees of freedom.

AMS SUBJECT CLASSIFICATIONS: Primary: 35Q30; 76D05; 35B41; 35B42; Secondary: 35K55; 35B65.

1. INTRODUCTION

We consider in this paper modifications of the Navier–Stokes equation in which hyperviscous terms are spectrally added (see (1.4) below). The motivation for our models comes from certain subgrid-scale (SGS) modeling techniques. In large-eddy simulation (LES) of turbulent incompressible flow, the divergence of a SGS tensor $S_{sg}(u)$ is added to the standard Navier–Stokes equations (NSE) to obtain the system

$$u_t + \operatorname{div}\left(S_{sg}(u)\right) + \nu A u + (u \cdot \nabla) u + \nabla p = f, \tag{1.1a}$$

$$\text{div } u = 0. \tag{1.1b}$$

Here $A = -\Delta$ while in 3D $u = (u_1, u_2, u_3)$ with $u_i = u_i\,(x,t)$, $g = (g_1, g_2, g_3)$ with $g_i = g_i\,(x,t)\,, i = 1, 2, 3$, and $p = p\,(x,t)$ where $x \in \Omega$, a domain in R^3. The unknown u is the velocity field of the fluid and g and p represent the external force and the pressure, respectively. The full SGS tensor takes the form of the stress tensor $S_{\text{sg}}\,(u) = \tau_{ij} = \widetilde{u_i u_j} - \widetilde{u_i}\widetilde{u_j}$ where in typical LES a low-pass filtering operation at scale δ is represented by the tilde. The most common approximate SGS model is the eddy-visocity (EV) method which assumes that $\tau_{ij}^d \equiv -\frac{1}{3}\tau_{kk}\delta^{ij} = 2\nu_T \widetilde{S_{ij}}$ where τ_{ij}^d is the deviatoric part of τ_{ij} and $\widetilde{S_{ij}}$ is the resolved strain-rate tensor. Generally in EV the eddy-viscosity acts equally on all scales of motion, but it has been shown in this case (see e.g. [6]) that the local energy flux can be poorly correlated with the local energy dissipation rate.

In [24] Kraichnan argued that the eddy viscosity should in fact depend on the wavenumber magnitude. Let C_{k} be the Kolmogorov constant, let $k_\delta \sim 1/\delta$ be the filter wavenumber, and let $E_<\,(k,t)$ be the energy spectrum of the filtered velocity field, then a working fit [9, 10] to the theoretical predictions of this dependence is

$$\nu_{\text{ev}}\,(k, k_\delta) = C_{\text{k}}^{-3/2}\left[\alpha_1 + \alpha_2 \exp\left(-3.03\frac{k_\delta}{k}\right)\right]\sqrt{\frac{E_<\,(k_\delta)}{k_\delta}} \tag{1.2}$$

where $a_1 = 0.441$, $a_2 = 15.3$. Typically k_δ is in the neighborhood of $1/l_\epsilon$, where l_ϵ is the Kolmogorov length scale. Among the salient features of the fit is the monotonically increasing behavior, with a relatively sharp rise as k climbs into the high wavenumber ranges. In LES of non-homogeneous flows, the implementation of spectral eddy-viscosity (SEV) is not easy. As advocated in [10], a possible approximation to the SEV term is to use instead a hyper-viscous term μA^α for a positive constant μ resulting in the hyperviscous Navier–Stokes equations (HNSE):

$$u_t + \mu A^\alpha u + \nu A u + (u \cdot \nabla)\,u + \nabla p = f, \tag{1.3a}$$

$$\text{div } u = 0. \tag{1.3b}$$

The HNSE has been used extensively in turbulence simulations. In particular Borue and Orsag [5, 6] were able to use (1.3) to produce an inertial range that is wider than with regular viscosity. For a more detailed review of these considerations concerning EV and SEV LES models, see e.g. [1, 7, 21] and the references contained therein. For further discussion of theoretical results concerning (1.3), see [3].

Another approximation to SEV is the spectral vanishing viscosity method (SVV), first introduced by Tadmor et al. [8, 30] as an alternative to the viscosity-solution (VS) method for conservation laws. Like EV, the viscosity-solution method employs a fixed differential operator resulting in "an uncontrollable process that can destroy the solution accuracy" [21]. The SVV method alternatively uses typically second-order viscosity kernels with wavenumber-dependent coefficients that reflect the behavior of (1.2), but which are not just small but zero for low wavenumbers. The original idea for conservation laws was to enforce the correct entropy dissipation as in VS while retaining spectral accuracy. In [21] the SVV method was applied in 3D turbulence simulations as an approximation to SEV and in particular to control high-wavenumber oscillations (in effect to enforce the Kolmogorov-predicted energy dissipation of high wavenumbers) while retaining spectral accuracy. Hyperviscous terms, though harder to implement, can be used in SVV via a discontinuous Galerkin approach.

In [1] we modified (1.3) by applying hyperviscosity spectrally, motivated by the SEV and SVV methodology. Let Ω be a periodic box; for simplicity assume $\Omega=(0,L)^3$. Then, "moding out" the constant vectors as in standard practice, A has eigenvalues $0<\lambda_1<\lambda_2<\ldots$ with corresponding eigenspaces $E_1, E_2, \ldots$. Let P_m be the projection onto $E_1\oplus E_2\oplus\ldots\oplus E_m$ and let $Q_m=I-P_m$, then we considered as here a class of operators $A_\varphi=\sum_{j=1}^{\infty} a\left(\lambda_j\right) E_j$ such that $A_\varphi \geqslant A_\mu = Q_m A^\alpha$ in the sense of quadratic forms, i.e. $\left(A_\varphi v, v\right) \geqslant \left(A_\mu v, v\right)$. An SVV-type operator, for example, is obtained for a distinguished class given by $a\left(\lambda_j\right)=0$ for $j\leqslant m_0\leqslant m$, $0\leqslant a\left(\lambda_j\right)\leqslant \lambda_j^\alpha$ for $m_0\leqslant j\leqslant m$, and $a_j\left(\lambda_j\right)\geqslant\lambda_j^\alpha$ for $j\geqslant m+1$, where $m_0\to\infty$ as $m\to\infty$. The operators A_φ are designed in particular to reflect SEV and SVV methodology and in particular the qualitative behavior of (1.2); since as in SVV we do not need a filtering mechanism k_δ is replaced by $1/l_\epsilon$.

The spectrally-hyperviscous NSE we study here as introduced in [1] are thus

$$u_t+\mu A_\varphi u+\nu A u+(u\cdot\nabla)u+\nabla p=f, \tag{1.4a}$$

$$\text{div } u=0. \tag{1.4b}$$

In [1] we demonstrated the global regularity of solutions to (1.4) and showed that, while higher-order bounds necessarily depend on m, this dependence is only as a fractional power of it for certain choices of α. In particular for $\alpha=5/2$ we obtained growth in the H^1-norm like $m^{1/2}$, and for $\alpha=2$ we obtained growth like $m^{7/12}$. We also showed that as $m\to\infty$ we have subsequence convergence to a weak Leray solution of the standard NSE.

In this paper we will focus on the finite-dimensional asymptotic character of the system (1.4). First, we will lay some groundwork and show that for $\alpha \geqslant 3/2$ we can obtain better H^β-estimates on the solution u than in [1]; the improved estimates also demonstrate the existence of absorbing sets that guarantee that the system (1.5) possesses a global attractor $\mathcal{A}$. Next we obtain scale-invariant estimates on the Hausdorff and fractal dimensions of $\mathcal{A}$ in terms of the Landau–Lifschitz theory of the number of degrees of freedom in turbulent flow. For typically-applicable A_φ in the distinguished class the estimates stay within the Landau–Lifschitz predictions for a range of m and m_0 large enough that at its upper end we heuristically expect virtual identification with NSE solutions. Next we show that for all A_φ the system (1.4) has an inertial manifold $\mathcal{M}$ of dimension $N > m$ if $\alpha > 5/2$. In the special class of A_φ such that $P_m A_\varphi = 0$ and $Q_m A_\varphi \geqslant Q_m A^\alpha$ we show that whenever $\alpha \geqslant 3/2$ we have a unique spectral-gap property that allows us to construct an inertial manifold $\mathcal{M}$ of dimension m if m is large enough. A similar result holds for large enough m_0 for most of the operators in the distinguished class that we envision will be used in practice, though for generally larger m and m_0 than in the special class above. We extend the latter result to a certain wider class by extending our spectral-gap arguments.

Our results for H^β-estimates and absorbing sets are discussed and established in Section 2. We now discuss our attractor results. Assume that f is time-independent, i.e., that $f = f(x)$, and let

$$L_f = \|f\|_2; \tag{1.5}$$

denote the fractal dimension of $\mathcal{A}$ by $\dim_{\mathrm{F}} \mathcal{A}$ and let $\dim_{\mathrm{H}} \mathcal{A}$ denote the Hausdorff dimension of $\mathcal{A}$.

We will express our primary attractor results in terms of the Kolmogorov length-scale l_ϵ and the Landau–Lifschitz estimates [27] of the number of degrees of freedom in turbulent flow (see e.g. [16, 33], and the references contained therein). Such estimates will give us useful information about the capability of (1.4) to approximate NSE dynamics.

Kolmogorov's mean rate of dissipation of energy in turbulent flow (see e.g. [16, 22, 33], and the references contained therein) is defined as

$$\epsilon = \lambda_1^{3/2} \nu \limsup_{T\to\infty} \int_0^T \left\| A^{1/2} u \right\|_2^2 ds. \tag{1.6}$$

The Kolmogorov length scale is

$$l_\epsilon = \left(\frac{\nu^3}{\epsilon}\right)^{1/4}. \tag{1.7}$$

Let $l_0 = \lambda_1^{-1/2}$; l_0 as so defined is a representation of characteristic macroscopic length, since $\lambda_1 \sim L^{-2}$. The foundational result for our attractor estimates is:

Theorem 1. *Let l_ϵ and l_0 be defined as above. Then for a constant K_α on the order of unity we have that for $\mu \geqslant \nu$*

$$\dim_{\mathrm{H}} A \leqslant \dim_{\mathrm{F}} A \leqslant K_\alpha \, [\lambda_m/\lambda_1]^{\frac{9(\alpha-1)}{10\alpha}} \left[\frac{l_0}{l_\epsilon}\right]^{\frac{6\alpha+9}{5\alpha}} \tag{1.8a}$$

and that for $\mu \leqslant \nu$

$$\dim_{\mathrm{H}} A \leqslant \dim_{\mathrm{F}} A \leqslant K_\alpha \left(\frac{\nu}{\mu}\right)^{\frac{9}{10\alpha}} [\lambda_m/\lambda_1]^{\frac{9(\alpha-1)}{10\alpha}} \left[\frac{l_0}{l_\epsilon}\right]^{\frac{6\alpha+9}{5\alpha}}. \tag{1.8b}$$

All bracketed constants and K_α are dimensionless, and depend on the shape (but not the size) of Ω and are thus scale-invariant. Because $\lambda_m \sim cm^{2/3}$ the growth in m of (1.8) is less than $m^{3/5}$. The growth in (1.8b) as μ gets smaller is as the relative size of ν to μ. Set

$$\lambda_m \leqslant \left(\frac{1}{l_\epsilon}\right)^2 \tag{1.9}$$

then since $2[9(\alpha-1)/(10\alpha)] + (6\alpha+9)/(5\alpha) = 3$ we have from (1.8) and (1.9) the following:

Theorem 2. *Let K_α, l_0, and l_ϵ be as above, then if* (1.10) *holds we have if $\mu \geqslant \nu$ that*

$$\dim_{\mathrm{H}} A \leqslant \dim_{\mathrm{F}} A \leqslant K_\alpha \left[\frac{l_0}{l_\epsilon}\right]^3 \tag{1.10a}$$

and if $\mu \leqslant \nu$ that

$$\dim_{\mathrm{H}} A \leqslant \dim_{\mathrm{F}} A \leqslant K_\alpha \left(\frac{\nu}{\mu}\right)^{\frac{9}{10\alpha}} \left[\frac{l_0}{l_\epsilon}\right]^3. \tag{1.10b}$$

This result holds independently of α and for K_α and ν/μ independent of m. Thus for $\mu \geqslant \nu$ we have straight-up agreement with the Landau–Lifschitz predictions $\sim[l_0/l_\epsilon]^3$ for the number of degrees of freedom in 3D turbulent flow even for very large m.

We now explore various choices of $\mu \leqslant \nu$. The Chapman–Enskog expansion suggests the choice $\mu \sim \nu^\alpha$ for (1.3); substituting this in (1.8b) gives

$$\dim_{\mathrm{H}} A \leqslant \dim_{\mathrm{F}} A \leqslant K_\alpha \left[(\lambda_m/\lambda_1)/\nu\right]^{\frac{9(\alpha-1)}{10\alpha}} \left[\frac{l_0}{l_\epsilon}\right]^{\frac{6\alpha+9}{5\alpha}} \tag{1.10c}$$

and we recover (1.10a) if we assume the modified condition $\lambda_m \leqslant \nu(1/l_\epsilon)^2$ or $(\lambda_m/\lambda_1) \leqslant \nu(l_0/l_\epsilon)^2$. By (2.15) below the Grashoff number $G = L_f/(\nu^2\lambda_1^{3/4})$ is an upper bound for $(l_0/l_\epsilon)^2$; assuming this is a sharp upper bound says that λ_m/λ_1 can be as large as $\lambda_m/\lambda_1 \sim \lambda_1^{-3/8} L_f^{1/2}(l_0/l_\epsilon)$. This would put λ_m past $1/l_\epsilon$ if $|\Omega| \sim c\lambda_1^{-3/2}$ and L_f are both larger than unity. Intuition from the Kolmogorov theory (see the corresponding remarks in the conclusion) and SEV/SVV results suggest that solutions of (1.4) in the distinguished-class case will be good NSE approximants if the wavenumbers up to $1/l_\epsilon$ only see standard NSE viscosity. For operators in the distinguished-class case such that m_0 is not too far from m this suggests that we have good NSE approximation while (1.10a) holds.

Since (1.3) is (1.4) with $m = 0$, our spectral considerations suggest that $\mu \sim \nu^\alpha$ is a lower bound for μ for $m > 0$. The observations in [7] and (1.2) for large k suggest setting $\mu \sim c\nu$ where $1/c$ is within α orders of magnitude of unity. An application of SVV methodology [21] uses the choice $\mu \geqslant \nu$. We discuss these choices in more detail in light of SVV and SEV considerations at the end of Section 3. The case $\mu \geqslant \nu$ is handled by (1.10a); the result of plugging the assumption $\mu \sim c\nu$ into (1.10b) under the condition (1.9) is:

Theorem 3. *Let K_α, l_0, c_α, and l_ϵ be as above, then if* (1.10) *holds and $c \leqslant 1$ we have for $c_\alpha = c^{-9/(10\alpha)}$ that*

$$\dim_{\mathrm{H}} A \leqslant \dim_{\mathrm{F}} A \leqslant c_\alpha K_\alpha \left[\frac{l_0}{l_\epsilon}\right]^3. \tag{1.10d}$$

Note that c_α is within an order of magnitude of unity. With (1.10a) and (1.10d) we have straight-up and nearly-straight-up agreement with the Landau–Lifschitz predictions $\sim[l_0/l_\epsilon]^3$ under the condition (1.9). Thus for λ_m as big as $(1/l_\epsilon)^2$ and if λ_{m_0} is not too far behind, e.g. $m_0 = m - b$ where $b < m/2$, we have (1.10a), (1.10d) for solutions that we would expect (by our remarks above regarding $1/l_\epsilon$) to be virtually indistinguishable from NSE solutions. In this sense we have globally-regular solutions that give "approximating" generalizations to 3D of the 2D results obtained in [11, 14, 31, 32] (see [16, 23] for background and further discussion).

Meanwhile, we would expect lower choices of λ_m, such as $(\lambda_m/\lambda_1) \leqslant (l_0/l_\epsilon)^{4/3}$, to still give good NSE approximation, but now with significantly lower powers on l_0/l_ϵ, showing potential of the model for such m and appropriate m_0 to reduce the number of degrees of freedom needed in practical simulation. Another choice for m_0 coming from the SVV methodology (see e.g. [21]) is $m_0 = 5m^{1/2}$ which by (1.9) and $\lambda_m \sim cm^{2/3}$ says that $\lambda_{m_0} \leqslant c_1(1/l_\epsilon)$ for some $c_1 > 1$. Though the approximation characteristics are not as robust, by the remarks above we still expect good NSE approximation while (1.10d) is satisfied.

Since the (dimensionless) Grashoff number $G = L_f/(\nu^2\lambda_1^{3/4})$ in 3D (see e.g. [16, 33], and the references contained therein) is an upper bound for $(l_0/l_\epsilon)^2$ by (2.15) below, expressing the above estimates in terms of G is straightforward. The term $[l_0/l_\epsilon]^{(6\alpha+9)/(5\alpha)}$ in (1.8) becomes $G^{(6\alpha+9)/(10\alpha)}$ and $[l_0/l_\epsilon]^3$ in (1.10) becomes $G^{3/2}$, while the condition (1.9) becomes $(\lambda_m/\lambda_1) \leqslant G$ and in the case $\mu \sim \nu^\alpha$ the condition $(\lambda_m/\lambda_1) \leqslant \nu(l_0/l_\epsilon)^2$ becomes $(\lambda_m/\lambda_1) \leqslant \nu G$. In the latter case this requires that $\lambda_m/\lambda_1 \leqslant \lambda_1^{-3/8} L_f^{1/2} G$ which is now definitely past $1/l_\epsilon$.

For the estimates of attractor dimension we will adapt the part of the "CFT" methodology (see e.g. [11, 14, 31, 32]), first developed for the 2D NSE, that relies both on trace formulas and the Lieb–Thirring inequality (LTI), the latter being first used in this context in [31, 32]. Related results for weak solutions of the NSE in 3D can be found in [4, 12, 13, 29]. In [15] for strong solutions of the model known variously as the NS-α, 3D LANS-α, or 3D Camassa–Holm equations, the CFT/LTI methodology is applied toward estimating attractor dimension and in the conclusion of this paper we will compare the attractor estimates developed in [15] with (1.10a–1.10d).

We now discuss results which use the Foias/Sell/Temam theory [17, 18, 33] to show that the system (1.4) possesses an inertial manifold, first for $\alpha > 5/2$ in the general case and then for $\alpha \geqslant 3/2$ for certain operators in the distinguished-class case if m is large enough. We recall the following definition from [17, 18]; also see [33]:

Definition 4 An inertial manifold $\mathcal{M}$ for (1.4) is a finite-dimensional manifold satisfying:

(i) $\mathcal{M}$ is Lipschitz
(ii) $\mathcal{M}$ is positively invariant for the semigroup, i.e. $S(t)\mathcal{M} \subset \mathcal{M}$ for all $t \geqslant 0$.
(iii) $\mathcal{M}$ attracts exponentially all the orbits of (1.4).

Here $S(t)$ is the mapping $S(t)u_0 = u(t)$ for each $u_0 \in H$. We have that S is well-defined for (1.4) for all $t \geqslant 0$ since in [3] we demonstrated that (1.3) has globally-regular (for $t > 0$) solutions for all $u_0 = u(0)$ in $H = PL^2(\Omega)$ whenever $\alpha > 5/4$, where P is the Leray projection; that this holds for (1.4) follows from the fact that $A_\varphi = A^\alpha - B_\alpha$ where B_α is a finite-dimensional (and therefore bounded) perturbation operator relative to A^α. Our first inertial-manifold existence result for (1.4) holds for all A_φ when $\alpha > 5/2$:

Theorem 5. *If* $\alpha > 5/2$ *then the conditions of Definition* 4 *are satisfied, so that the system* (1.4) *has an inertial manifold* $\mathcal{M}$*.*

Theorem 5 turns out to follow fairly straightforwardly from the theory developed in the celebrated results of Foias et al. [17, 18], as presented in [33]. These works introduced the concept of an inertial manifold and demonstrated the existence of inertial manifolds for a wide variety of semilinear evolutionary systems.

The salient feature of $\mathcal{M}$ is that $\mathcal{M}$ is a graph over $P_N H$. We need the spectral-gap properties of the hyperviscous part of A_φ to prove Theorem 5 for the entire class of A_φ, thus we need to assume in Theorem 5 that $N > m$. To reinforce the idea that we have a good closure model for the NSE, we want an $\mathcal{M}$ such that $\mathcal{M}$ is a graph over $P_m H$. For large enough m this would say, at least for certain distinguished-class A_φ, that trajectories on $\mathcal{M}$ are controlled by essentially NSE dynamics. By getting into some of the details of the Foias/Sell/Temam proof we can exploit a unique spectral-gap property to establish the following; note that it holds for a wider range of α than Theorem 5.

Theorem 6. *Let* A_φ *satisfy* $P_m A_\varphi = 0$ *and* $Q_m A_\varphi \geqslant Q_m A^\alpha$ *with* $\alpha \geqslant 3/2$*. Then for* m *large enough the system* (1.4) *has an inertial manifold* $\mathcal{M}$ *of dimension* m*.*

In proving Theorem 6 we will obtain fairly explicit estimates for the dimension of $\mathcal{M}$ which will be particularly explicit for $\alpha \geqslant 5/2$. Meanwhile, the operators in the distinguished class that we envision will be typically applied in practice are of the form

$$a(\lambda_j) \geqslant (\mu_j/\mu)\lambda_j^\alpha, \quad m_0 + 1 \leqslant j \leqslant m \tag{1.11}$$

with $\{\mu_j\}_{j=m_0+1}^m$ a monotonically-increasing sequence such that $0 < \mu_j \leqslant \mu$. Such operators have basically the same spectral-gap property as the operator in Theorem 6, but with the gap centered at m_0 rather than at m. Thus a simple corollary of Theorem 6 is the following inertial-manifold result for this class of A_φ:

Theorem 7. *Let A_φ be in the distinguished class with $\alpha \geqslant 3/2$ and such that* (1.11) *is satisfied. Then for m_0 large enough the system* (1.4) *has an inertial manifold $\mathcal{M}$ of dimension m_0.*

Since the size of m and m_0 in Theorems 6 and 7 will be seen generally to depend respectively on $1/\mu$ and $1/\mu_{m_0+1}$, we expect that we need a significantly larger "NSE" part of $\mathcal{M}$ in Theorem 7 than in Theorem 6. In both cases, for large enough m (respectively m_0), we have an inertial manifold for a system in which the inertial range essentially behaves according to standard NSE physics, and in particular trajectories on $\mathcal{M}$ are controlled by essentially NSE dynamics.

We can generalize Theorem 7 further by considering a class of operators A_φ in which for an integer m_1 between m_0 and m and a monotonically-increasing sequence of μ_j with $0 < \mu_j \leqslant \mu$ we have that $a(\lambda_j) = (\mu_j/\mu)\lambda_j^\alpha$ for $m_1 + 1 \leqslant j \leqslant m$ and such that $a(\lambda_j) = (\mu_j/\mu)\lambda_j^{\eta_j \alpha}$ for $m_0 + 1 \leqslant j \leqslant m_1$ where $0 < \eta_j < 1/2$. By the last condition on η_j we again will have a spectral gap, now centered at m_1. The following then follows also as a simple corollary of Theorem 6, and with it we finish stating our main results:

Theorem 8. *Let A_φ, m_1, μ_j, and η_j be as above with $\alpha \geqslant 3/2$. Then for m_1 large enough the system* (1.4) *has an inertial manifold $\mathcal{M}$ of dimension m_1.*

Section 2 below lays out some preliminary observations and obtains relevant a priori estimates, including improved H^β-estimates and absorbing-set estimates needed for the attractor and inertial manifold theory. Section 3 proves the estimate (1.8) from which we will show that the attractor results in (1.10a–1.10d) follow. Section 4 establishes the inertial manifold results.

2. PRELIMINARIES, A PRIORI ESTIMATES, AND ABSORBING SETS

We express the Sobolev inequalities on Ω in terms of the operator $A = -\Delta$:

$$\|v\|_q \leqslant M_1 \left\|A^\gamma v\right\|_p \tag{2.1}$$

where $q \leqslant 3p/(3-2\gamma p)$ and $M_1 = M_1(\gamma, p, q, \Omega)$. For the semigroup $\exp(-tA)$ we have the decay estimate

$$\left\| e^{-tA} v \right\|_2 \leqslant \|v\|_2 \, e^{-\lambda_1 t} \tag{2.2}$$

and, since A is analytic there is a constant c_2 such that

$$\left\| A^{\beta} e^{-tA} v \right\|_2 \leqslant c_2 t^{-\beta} \|v\|_2 \tag{2.3}$$

for any $\beta > 0$ where A^{β} is defined by $A^{\beta} = \sum_{n=1}^{\infty} \lambda_n^{\beta} E_n$ where as above E_n is the projection onto the nth eigenspace. Like the standard NSE, (1.4) satisfies an energy inequality, which we derive as follows: taking the inner product of both sides of (1.4) with u we have that

$$\frac{1}{2}\frac{d}{dt} \|u\|_2^2 + \nu \left\| A^{1/2} u \right\|_2^2 + \mu \left\| A_{\varphi}^{1/2} u \right\|_2^2 = (f, u) \tag{2.4}$$

noting that since div $u = 0$ we have that $(\nabla p, u) = 0$ and $((u \cdot \nabla) u, u) = -((\text{div } u) u, u) = 0$. Now

$$(f, u) = \left(A^{-1/2} f, A^{1/2} u \right) \leqslant \frac{\nu}{2} \left\| A^{1/2} u \right\|_2^2 + \frac{1}{2\nu} \left\| A^{-1/2} f \right\|_2^2 \tag{2.5}$$

and combining (2.5) with (2.4) and multiplying by 2 we have our basic energy inequality

$$\frac{d}{dt} \|u\|_2^2 + \nu \left\| A^{1/2} u \right\|_2^2 + 2\mu \left\| A_{\varphi}^{1/2} u \right\|_2^2 \leqslant \frac{1}{\nu \lambda_1} \|f\|_2^2 \tag{2.6}$$

where we note that by Poincaré's inequality $\left\| A^{-1/2} f \right\|_2 \leqslant \lambda_1^{-1/2} \|f\|_2$; note that (2.6) reduces to the standard NSE energy inequality when $\mu = 0$. We will use 2 consequences of (2.6), the first obtained by discarding the term $\mu \left\| A_{\varphi}^{1/2} u \right\|_2^2$ and again using Poincaré to obtain

$$\frac{d}{dt} \|u\|_2^2 + \nu \lambda_1 \|u\|_2^2 \leqslant \frac{1}{\nu \lambda_1} \|f\|_2^2 \tag{2.7}$$

so that, setting

$$L_f = \sup_{t \geqslant 0} \|f\|_2^2 \tag{2.8}$$

we have that

$$\frac{d}{dt} \|u\|_2^2 + \nu \lambda_1 \|u\|_2^2 \leqslant \frac{L_f^2}{\nu \lambda_1}. \tag{2.9}$$

Solving the differential inequality (2.9) we have that for $u_0 = u(x,0)$

$$\|u(t)\|_2^2 \leqslant \|u_0\|_2^2 e^{-\nu\lambda_1 t} + \int_0^t \left(\frac{L_f^2}{\nu\lambda_1}\right) e^{-\nu\lambda_1(t-s)} ds \tag{2.10}$$

or, since $L_f^2/(\nu\lambda_1)$ is a constant,

$$\|u(t)\|_2^2 \leqslant \|u_0\|_2^2 e^{-\nu\lambda_1 t} + \left(\frac{L_f}{\nu\lambda_1}\right)^2. \tag{2.11}$$

Thus, we have the a priori estimate

$$\|u(t)\|_2^2 \leqslant \|u_0\|_2^2 + \left(\frac{L_f}{\nu\lambda_1}\right)^2 \equiv U_{L_f}^2 \tag{2.12}$$

and we also have from (2.11) that

$$\limsup_{t\to\infty} \|u(t)\|_2 \leqslant \frac{L_f}{\nu\lambda_1}. \tag{2.13}$$

Note that (2.13) implies that for any $\delta > 0$ the ball of radius $(1+\delta)$ $L(\nu\lambda_1)^{-1}$ is an absorbing set in H for all trajectories $\bigcup_{t\geqslant 0} u(t)$.

The second way we use (2.6) is to again discard the term $\mu \left\| A_\varphi^{1/2} u \right\|_2$ and integrate from 0 to T to obtain

$$\|u(T)\|_2^2 + \nu \int_0^T \left\| A^{1/2} u \right\|_2^2 ds \leqslant \|u_0\|_2^2 + \int_0^T \left(\frac{L_f^2}{\nu\lambda_1}\right) ds \tag{2.14}$$

from which we obtain

$$\limsup_{t\to\infty} \frac{1}{T} \int_0^T \left\| A^{1/2} u \right\|_2^2 ds \leqslant \frac{L_f^2}{\nu^2\lambda_1} \tag{2.15}$$

which verifies that the left-hand side is finite and gives a useful upper bound for Theorem 1.

Next, we obtain higher-order a priori estimates. We decompose u as $u = P_m u + Q_m u$; on $P_m H$ the operator A^β is bounded for any $\beta > 0$, and we have that from (2.12) and the semigroup property of solutions that for $t \geqslant \tau \geqslant 0$

$$\begin{aligned} \left\| A^\beta P_m u(t) \right\|_2 &\leqslant \left\| A^\beta \right\|_{2;P_m H} \|u(t)\|_2 \\ &\leqslant \lambda_m^\beta \left[\|u(\tau)\|_2 e^{-\nu\lambda(t-\tau)} + \left(\frac{L_f}{\nu\lambda_1}\right)^2 \right]^{1/2} \end{aligned} \tag{2.16}$$

so that for a fixed $\tau > 0$

$$\left\| A^{\beta} P_m u(t) \right\|_2 \leqslant \lambda_m^{\beta} \left[\|u(\tau)\|_2 \, e^{-\nu\lambda(t-\tau)} + \left(\frac{L_f}{\nu\lambda_1} \right)^2 \right]^{1/2} \tag{2.17}$$

for all $t \geqslant \tau$.

Without loss of generality we can assume that $A_{\varphi} = A_{\mu} = Q_m A^{\alpha} = A^{\alpha}$ on $Q_m H$, so that for $t \geqslant \tau$ we have that $Q_m u$ satisfies the integral equation

$$Q_m u(t) = Q_m e^{-\nu(\tau-\tau)A} e^{-\mu(t-\tau)A^{\alpha}} u(\tau) + \int_{\tau}^{t} e^{-\nu(t-s)A} e^{-\mu(t-s)A^{\alpha}} Q_m P f(s) \, ds$$

$$+ \int_{\tau}^{t} e^{-\nu(t-s)A} e^{-\mu(t-s)A^{\alpha}} Q_m P (u(s) \cdot \nabla) u(s) \, ds \tag{2.18}$$

where P is the Leray projection; we have used the fact that A and A^{α} commute. Applying A^{β} to both sides of (2.18), noting that Q_m and P commute with A, and writing $e^{-\mu t A^{\alpha}} = \left(e^{-\mu(t/2)A^{\alpha}}\right)^2$, we have that

$$\begin{aligned}
\left\| A^{\beta} Q_m u(t) \right\|_2 &\leqslant \left\| A^{\beta} e^{-\mu(t-\tau)A^{\alpha}} Q_m u(\tau) \right\|_2 e^{-\nu\lambda_1(t-\tau)} \\
&\quad + \int_{\tau}^{t} \left\| A^{\beta} e^{-\mu(t-s)A^{\alpha}} f(s) \right\|_2 ds \\
&\quad + \int_{\tau}^{t} \left\| A^{\beta} e^{-\mu(t-s)A^{\alpha}} (u \cdot \nabla) u \right\|_2 ds \\
&\leqslant \mu^{-\beta/\alpha} \left\| (\mu A^{\alpha})^{\beta/\alpha} \left[e^{-\frac{(t-\tau)}{2}(\mu A^{\alpha})} \right]^2 u(\tau) \right\|_2 \\
&\quad + \mu^{-\beta/\alpha} \int_{\tau}^{t} \left\| (\mu A^{\alpha})^{\beta/\alpha} \left[e^{-\frac{(t-s)}{2}(\mu A^{\alpha})} \right]^2 f(s) \right\|_2 ds \\
&\quad + \mu^{-\beta/\alpha} \int_{\tau}^{t} \left\| (\mu A^{\alpha}) \left[e^{-\frac{(t-s)}{2}(\mu A^{\alpha})} \right]^2 (u(s) \cdot \nabla) u(s) \right\|_2 ds.
\end{aligned} \tag{2.19}$$

Now from (2.1) there is a constant $M_2 = M_1(3/4, 1, 2, \Omega)$ such that $\|v\|_2 \leqslant M_2 \left\| A^{3/4} v \right\|_1$, and note that $(u \cdot \nabla) u = \text{div } (u \otimes u)$ for the appropriate tensor product $u \otimes u$. Also note that $A^{-1/2}\text{div}$ commutes with A and is a bounded operator on H of norm $\leqslant 1$. Writing div $(u \otimes u)$ as $A^{1/2}\left(A^{-1/2}\text{div } (u \otimes u)\right)$ and using the Sobolev inequality, we have using (2.2) and (2.3) that

$$
\begin{aligned}
\left\|A^{\beta} Q_m u(t)\right\|_2 &\leqslant \mu^{-\beta/\alpha} \left\|\left(\mu A^{\alpha}\right)^{\beta/\alpha} e^{-\left(\frac{t-\tau}{2}\right)\mu A^{\alpha}} u(\tau)\right\|_2 e^{-\lambda_m^{\alpha}(\mu/2)(t-\tau)} \\
&\quad + \mu^{-\beta/\alpha} \int_{\tau}^{t} \left\|\left(\mu A^{\alpha}\right)^{\beta/\alpha} e^{-\left(\frac{t-s}{2}\right)\mu A^{\alpha}} f(s)\right\|_2 e^{-\lambda_m^{\alpha}(\mu/2)(t-s)} ds \\
&\quad + \mu^{-\frac{\beta+1/2}{\alpha}} \int_{\tau}^{t} \left\|\left(\mu A^{\alpha}\right)^{\frac{\beta+1/2}{\alpha}} e^{-\left(\frac{t-s}{2}\right)\mu A^{\alpha}} A^{-1/2} \operatorname{div}(u \otimes u)\, e^{-\lambda_m(\mu/2)(t-s)}\right\|_2 ds \\
&\leqslant c_2 \mu^{-\beta/\alpha} (t-\tau)^{-\beta/\alpha} 2^{\beta/\alpha} \|u(\tau)\|_2 e^{-\lambda_m(\mu/2)(t-\tau)} \\
&\quad + 2^{\beta/\alpha} c_2 \mu^{-\beta/\alpha} \int_{\tau}^{t} \frac{e^{-\lambda_m^{\alpha}(\mu/2)(t-s)}}{(t-s)^{\beta/\alpha}} \|f(s)\|_2 ds \\
&\quad + \mu^{-\frac{\beta+1/2}{\alpha}} \int_{\tau}^{t} \left\|A^{-1/2} \operatorname{div}\left(\mu A^{\alpha}\right)^{\frac{\beta+1/2}{\alpha}} e^{-\frac{(t-s)}{2}\mu A^{\alpha}} u \otimes u\right\|_2 e^{-\lambda_m(\mu/2)(t-s)} ds \\
&\leqslant c_2 (2/\mu)^{\beta/\alpha} \|u(\tau)\|_2 \frac{e^{-\lambda_m^{\alpha}(\mu/2)(t-\tau)}}{(t-\tau)^{\beta/\alpha}} \\
&\quad + c_2 (2/\mu)^{\beta/\alpha} L_f \int_{\tau}^{t} \frac{e^{-\lambda_m^{\alpha}(\mu/2)(t-s)}}{(t-s)^{\beta/\alpha}} ds \\
&\quad + \mu^{-(\beta+5/4)/\alpha} \int_{\tau}^{t} \left\|\left(\mu A^{\alpha}\right)^{\frac{\beta+5/4}{\alpha}} e^{-(t-s)(\mu/2)A^{\alpha}} \right. \\
&\qquad \left. \left(A^{-3/4} u \otimes u\right)\right\|_2 e^{-\lambda_m(\mu/2)(t-s)} ds \\
&\leqslant c_2 (2/\mu)^{\beta/\alpha} \|u(\tau)\|_2 \frac{e^{-\lambda_m^{\alpha}(\mu/2)(t-\tau)}}{(t-\tau)^{\beta/\alpha}} + c_2 (2/\mu)^{\beta/\alpha} L_f \int_{\tau}^{t} \frac{e^{-\lambda_m^{\alpha}(\mu/2)(t-s)}}{(t-s)^{\beta/\alpha}} ds \\
&\quad + c_2 (2/\mu)^{\frac{\beta+5/4}{\alpha}} \int_{\tau}^{t} \frac{e^{-\lambda_m^{\alpha}(\mu/2)(t-s)}}{(t-s)^{\frac{\beta+5/4}{\alpha}}} \left\|A^{-3/4}(u \otimes u)\right\|_2 ds \\
&\leqslant c_2 (2/\mu)^{\beta/\alpha} \left[\|u(\tau)\| \frac{e^{-\lambda_m^{\alpha}(\mu/2)(t-\tau)}}{(t-\tau)^{\beta/\alpha}} + L_f \int_{\tau}^{t} \frac{e^{-\lambda_m^{\alpha}(\mu/2)(t-s)}}{(t-s)^{\beta/\alpha}} ds\right] \\
&\quad + c_2 (2/\mu)^{\frac{\beta+5/4}{\alpha}} M_2 \int_{\tau}^{t} \frac{e^{-\lambda_m^{\alpha}(\mu/2)(t-s)}}{(t-s)^{\frac{\beta+5/4}{\alpha}}} \|u \otimes u\|_1 ds. \qquad (2.20)
\end{aligned}
$$

Now for any $0<\gamma<1$

$$\begin{aligned}\int_{\tau}^{t}\frac{e^{-\lambda_m^{\alpha}(\mu/2)(t-s)}}{(t-s)^{\gamma}}ds &\leqslant \int_{0}^{\infty}\frac{e^{-\lambda_m^{\alpha}(\mu/2)s}}{s^{\gamma}}ds\\ &\leqslant (1-\gamma)^{-1}e^{-\gamma}\left[\lambda_m^{\alpha}(\mu/2)\right]^{\gamma-1}\end{aligned} \tag{2.21}$$

(see [2] or [18]). So, combining this with (2.21) we have that for $t>\tau$

$$\begin{aligned}\left\|A^{\beta}Q_m u(t)\right\|_2 \leqslant c_2(2/\mu)^{\beta/\alpha}&\left[\|u(\tau)\|_2\frac{e^{-\lambda_m^{\alpha}(\epsilon/2)(t-\tau)}}{(t-\tau)^{\beta/\alpha}}\right.\\ &\left.+L_f\left(1-\frac{\beta}{\alpha}\right)^{-1}e^{-\beta/\alpha}\left(\frac{2}{\mu}\right)^{1-\beta/\alpha}\frac{1}{\lambda_m^{\alpha-\beta}}\right]\\ &+c_2M_2\sup_{s\geqslant\tau}\|u(s)\|_2^2(1-\gamma)^{-1}e^{-\gamma}\frac{2}{\mu}\left[\frac{1}{\lambda_m}\right]^{\alpha-(\beta+5/4)}\end{aligned} \tag{2.22}$$

where $\gamma=(\beta+5/4)/\alpha$.

Note that we have implicitly assumed that $\beta+5/4<\alpha$. To simply get a priori bounds, we can set $\tau=0$ and use $\|u(s)\|_2\leqslant U_{L_f}$ for all $s\geqslant 0$. To show the existence of absorbing sets in $H^{2\beta}$, choose τ large enough so that, using (2.11), $\|u(s)\|_2\leqslant\left[(1+\delta_0)L_f\right]/(\nu\lambda_1)$ for all $s\geqslant\tau$, then we have from (2.23), using $\lambda_m\sim c\lambda_1 m^{2/3}$, and simplifying, that

$$\begin{aligned}\limsup_{t\to\infty}\left\|A^{\beta}Q_m u(t)\right\|_2 &\leqslant c_2\left[\frac{\alpha}{\alpha-\beta}e^{-\beta/\alpha}\right]\left(\frac{2}{\mu}\right)\frac{1}{\lambda_m^{\alpha-\beta}}L_f\\ &\quad+c_2M_2(1+\delta_0)^2\left[\frac{\alpha}{\alpha-(\beta+5/4)}e^{-(\beta+5/4)/\alpha}\right]\\ &\quad\times\left(\frac{2}{\mu}\right)\frac{1}{\lambda_m^{\alpha-(\beta+5/4)}}\left(\frac{L_f}{\nu\lambda_1}\right)^2\\ &=c_2\left[\frac{\alpha}{(\alpha-\beta)e^{\beta/\alpha}}\right]\frac{2\lambda_1^{1+\beta-\alpha}}{c^{\alpha-\beta}m^{2(\alpha-\beta)/3}}\left(\frac{\nu}{\mu}\right)\frac{L_f}{\nu\lambda_1}\\ &\quad+c_2M_2(1+\delta_0)^2\left[\frac{\alpha}{[\alpha-(\beta+5/4)]e^{(\beta+5/4)/\alpha}}\right]\\ &\quad\times\left(\frac{2}{\mu}\right)\frac{1}{\lambda_m^{\alpha-(\beta+5/4)}}\left(\frac{L_f}{\nu\lambda_1}\right)^2\\ &\equiv K_1^{\alpha,\mu,m}\frac{L_f}{\nu\lambda_1}+K_2^{\alpha,\mu,m}\left(\frac{L_f}{\nu\lambda_1}\right)^2.\end{aligned} \tag{2.23}$$

Note that the powers on λ_m or m in the denominators of the right-hand side of (2.23) are positive, so that $K_i^{\alpha,\mu,m}$ decreases as m increases, $i=1,2$. In particular, since it is safe to assume that $\lambda_m \geqslant 1$, we have that $K_i^{\alpha,\mu,m} \leqslant K_i^{\alpha,\mu}$ for constants $K_i^{\alpha,\mu}$ independent of m. Combining with (2.17), we have that

$$\limsup_{t\to\infty} \left\| A^{\beta} u(t) \right\|_2^2 \leqslant (\lambda_m^{2\beta} + K_1^{\alpha,\mu,m}) \frac{L_f}{\nu\lambda_1} + K_2^{\alpha,\mu,m} \left(\frac{L_f}{\nu\lambda_1} \right)^2 \equiv \rho_{m,\beta} \quad (2.24)$$

so that the ball of radius $\rho_\beta = (1+\delta)\left(\rho_{m,\beta}\right)^{\frac{1}{2}}$ in $PH^{2\beta}$ is an absorbing set for all trajectories.

Note again that in (2.23) we obtain an a priori bound on $\left\| A^{\beta} u(t) \right\|_2$ for all $t > 0$ by setting $\tau = 0$ and by replacing $\sup_{s \geqslant \tau} \| v(s) \|_2$ by U_{L_f}; if $u(0) = u_0$ is in $D\left(A^{\beta}\right)$, the domain of A^{β}, then we can remove the coefficient $(t-\tau)^{-\beta/\alpha} = t^{-\beta/\alpha}$. We have noted that we need $\beta + 5/4 < \alpha$, which says that $\beta < 1/4$ for $\alpha = 3/2$, $\beta < 3/4$ for $\alpha = 2$, $\beta < 5/4$ for $\alpha = 5/2$, $\beta < 7/4$ for $\alpha = 3$, $\beta < 9/4$ for $\alpha = 4$, and so on. Note also the connection with the Kolmogorov theory in that the right-hand side of (2.22) is small for large m. In particular, again using $\lambda_m \sim c\lambda_1 m^{2/3}$, we have that $\lambda_m^{\alpha-(\beta+5/4)} \sim (c\lambda_1)^{\alpha-(\beta+5/4)} m^{2[\alpha-(\beta+5/4)]/3}$; if e.g. $\beta = 1/2$ we have that $2[\alpha - (\beta + 5/4)]/3 \geqslant 1$ when $2(\alpha - 7/4) \geqslant 3$ or $\alpha \geqslant 13/4$. Thus, when, say, $\alpha > 3$, the right-hand side of (2.22) is significantly small. Note further that as long as $\alpha > 7/4$ we can have $\beta \geqslant 1/2$ in (2.22) and (2.23); to get estimates on $\left\| A^{\beta} u(t) \right\|_2$ for $\beta \geqslant 1/2$ when $3/2 \leqslant \alpha \leqslant 7/4$, we need to bootstrap the estimate (2.22). Standard techniques can do this, of course, but the estimates will be a bit messy, so we omit the details here.

We also note that (2.22) improves our estimates in [1]: when $\beta = 1/2$ we have overall growth in m like $m^{1/3}$, rather than $m^{1/2}$ for $\alpha = 5/2$ and $m^{7/12}$ for $\alpha = 2$, and here this growth is as $m^{1/3}$ for all α. This concludes our discussion of preliminary results.

3. PROOF OF THE ATTRACTOR ESTIMATES

We follow the development in ([33], Chapters V and VI); if we write (1.5) as

$$\frac{du(t)}{dt} = F(u(t)), \quad t > 0 \quad (3.1a)$$

$$u(0) = u_0 \quad (3.1b)$$

with solution $S(t): u_0 \in H \to u(t) \in H$ then the linearized problem is

$$\frac{dU(t)}{dt} = F'[S(t)u_0] \cdot U(t) \tag{3.2a}$$

$$U(0) = \xi \in H. \tag{3.2b}$$

For u_0 fixed in H, let $\xi_1, \dots, \xi_M$ be M elements of H and let $U_1, \dots, U_M$ be the corresponding solutions of (3.2). Let $q_M = q_M(t, u_0; \xi_1, \dots, \xi_M)$ be the projection $q_M H = span\{U_1, \dots, U_M\}$, and let $\varphi_1(t), \dots, \varphi_M(t)$ be an orthonormal basis for $q_M(t)H$. We need to find M so that uniformly in space and asymptotically in time

$$\begin{aligned} 0 &\geqslant \mathrm{Tr} F'(S(t)u_0) \circ q_M(t) \\ &= \sum_{j=1}^{\infty} \left(\mathrm{Tr} F'(u(t)) \circ q_M(t)\varphi_j(t), \varphi_j(t)\right) \\ &= \sum_{j=1}^{m} \left(F'(u(t))\varphi_j(t), \varphi_j(t)\right). \end{aligned} \tag{3.3}$$

Now

$$\begin{aligned} \left(F'(u)\cdot\varphi_j, \varphi_j\right) = &-\nu\left(A\varphi_j, \varphi_j\right) - \mu\left(A_\varphi \varphi_j, \varphi_j\right) \\ &- \left(\left(\varphi_j \cdot \nabla\right)u, \varphi_j\right) \leqslant -\nu\left(A\varphi_j, \varphi_j\right) - \mu\left(Q_m A^\alpha \varphi_j, \varphi_j\right) \\ &- \left(\left(\varphi_j \cdot \nabla\right)u, \varphi_j\right) \text{ and} \end{aligned}$$

$$\left|\sum_{j=1}^{M}\left(\left(\varphi_j \cdot \nabla\right)u, \varphi_j\right)\right| \leqslant \int_\Omega \left|\sum_{j=1}^{M}\sum_{i,k=1}^{3} \varphi_{ji}(x) D_i u_k(x) \varphi_{jk}(x)\right| dx. \tag{3.4}$$

For each x we have that

$$\left|\sum_{j=1}^{M}\sum_{i,k=1}^{3} \varphi_{ji}(x) D_i u_k(x) \varphi_{jk}(x)\right| \leqslant |Du(x)|\,\rho(x) \tag{3.5}$$

where

$$|Du(x)| = \left\{\sum_{j,k=1}^{3} |D_i u_k(x)|^2\right\}^{1/2} \tag{3.6}$$

and

$$\rho(x) = \sum_{i=1}^{3}\sum_{j=1}^{M}\left(\varphi_{ji}(x)\right)^2. \tag{3.7}$$

Combining the above observations with (3.3) we have that

$$\mathrm{Tr}F'(S(t)u_0)\circ q_m(t) \leqslant -\nu\sum_{j=1}^{M}\left\|A^{1/2}\varphi_j(t)\right\|_2^2 -\mu\sum_{j=1}^{M}\left\|Q_m A^{\alpha/2}\varphi_j(t)\right\|_2^2+\int_\Omega |Du|\,\rho dx. \quad (3.8)$$

We now prepare to use the generalized form of the Lieb–Thirring inequality in dimension $n=3$ as developed in [33]: let $a(v,u)$ be a coercive quadratic form of order m_0 then for e.g. φ_j as above we have:

Theorem 9. (Lieb–Thirring Inequality)

$$\left(\int_\Omega \rho(x)^{q/(q-1)}\,dx\right)^{2m_0(q-1)/3} \leqslant K_1\sum_{j=1}^{M} a\left(\varphi_j,\varphi_j\right) \quad (3.9)$$

for all $q\in\left(\max\left\{1,\frac{3}{2m_0}\right\},1+\frac{3}{2m_0}\right)$ *and where* K_1 *depends on* m_0, p, *and* q, *and on the shape (but not the size) of* Ω.

The quadratic form we will use is $a(v,u)=(A^\alpha v,u)=\left(A^{\alpha/2}v,\,A^{\alpha/2}u\right)$ so that the order of our quadratic form is $m_0=\alpha$. We have that Theorem 6 holds provided that $q=1+3/(2\alpha)$ so that $p=q/(q-1)=(2\alpha)/3+1=(2\alpha+3)/3$ where $p^{-1}+q^{-1}=1$. Note that $q=2$ when $\alpha=3/2$, obtaining the 3D analog of (3.9) for the 2D case (where we substitute 2 for 3 in (3.9)). Note also that $2m_0(q-1)/3=[2\alpha(3/(2\alpha))]/3=1$. For $q=1+3/(2\alpha)$ we apply Young's inequality in the form

$$ab\leqslant \epsilon_p a^p+c_{\epsilon_p}b^q \quad (3.10)$$

where

$$c_{\epsilon_p}=\frac{p-1}{p^q\epsilon_p^{1/(p-1)}} \quad (3.11)$$

to obtain that

$$\int_\Omega |Du|\,\rho dx \leqslant \|\rho\|_p\,\|Du\|_q \leqslant \epsilon_p\,\|\rho\|_p^p+c_{\epsilon_p}\,\|Du\|_q^q \quad (3.12)$$

where ϵ_p is to be chosen later.

Applying (3.9–3.11) to (3.8), we have (with $q/(q-1)=p$) that

$$\begin{aligned}
\mathrm{Tr} F'(S(t)u_0)\circ q_M(t) &\leqslant -\nu\sum_{j=1}^{M}\left\|A^{1/2}\varphi_j(t)\right\|_2^2 \\
&\quad -\mu\sum_{j=1}^{M}\left\|Q_m A^{\alpha/2}\varphi_j(t)\right\|_2^2+\int_\Omega |Du|\,\rho dx \\
&\leqslant -\nu\sum_{j=1}^{M}\left\|A^{\alpha/2}\varphi_j(t)\right\|_2^2-\mu\sum_{j=1}^{M}\left\|Q_m A^{\alpha/2}\varphi_j(t)\right\|_2^2 \\
&\quad +\epsilon_p\int_\Omega \rho^p dx+c_{\epsilon_p}\|Du\|_q^q \\
&\leqslant -\nu\sum_{j=1}^{M}\left\|A^{1/2}\varphi_j(t)\right\|_2^2-\mu\sum_{j=1}^{M}\left\|Q_m A^{\alpha/2}\varphi_j(t)\right\|_2^2 \\
&\quad +\epsilon_p K_1\sum_{j=1}^{M}\left\|A^{\alpha/2}\varphi_j(t)\right\|_2^2+c_{\epsilon_p}\|Du\|_q^q. \qquad (3.13)
\end{aligned}$$

Now $\|A^{\alpha/2}\varphi_j(t)\|_2^2=\|P_m A^{\alpha/2}\varphi_j(t)\|_2^2+\|Q_m A^{\alpha/2}\varphi_j(t)\|_2^2$ and we have that

$$\begin{aligned}
\left\|P_m A^{\alpha/2}\varphi_j(t)\right\|_2^2 &\leqslant \left\|P_m A^{\frac{\alpha-1}{2}}\right\|_2^2\left\|A^{1/2}\varphi_j(t)\right\|_2^2 \\
&\leqslant \lambda_m^{\alpha-1}\left\|A^{1/2}\varphi_j(t)\right\|_2^2. \qquad (3.14)
\end{aligned}$$

We first assume that

$$\mu\leqslant\nu \quad \text{and} \quad \lambda_m^{(\alpha-1)}\geqslant 1 \qquad (3.15)$$

so that, choosing

$$\epsilon_p=\frac{\mu}{2\lambda_m^{\alpha-1}K_1} \qquad (3.16)$$

we have, with (3.14) and (3.15),

$$\begin{aligned}
&\epsilon_p K_1\sum_{j=1}^{M}\left\|A^{\alpha/2}\varphi_j(t)\right\|_2^2 \\
&\leqslant\left(\frac{\mu}{2\lambda_m^{\alpha-1}K_1}\right)K_1\left[\lambda_m^{\alpha-1}\sum_{j=1}^{M}\left(\left\|P_m A^{1/2}\varphi_j(t)\right\|_2^2+\left\|Q_m A^{\alpha/2}\varphi_j(t)\right\|_2^2\right)\right]
\end{aligned}$$

$$\leqslant \left(\frac{\mu}{2\lambda_m^{\alpha-1}}\right)\lambda_m^{\alpha-1}\left[\sum_{j=1}^{M}\left\|A^{1/2}\varphi_j(t)\right\|_2^2+\sum_{j=1}^{M}\left\|Q_m A^{\alpha/2}\varphi_j(t)\right\|_2^2\right]$$
$$\leqslant \frac{\mu}{2}\left[\sum_{j=1}^{M}\left\|A^{1/2}\varphi_j(t)\right\|_2^2+\sum_{j=1}^{M}\left\|Q_m A^{\alpha/2}\varphi_j(t)\right\|_2^2\right]$$
$$\leqslant \frac{\nu}{2}\sum_{j=1}^{M}\left\|A^{1/2}\varphi_j(t)\right\|_2^2+\frac{\mu}{2}\sum_{j=1}^{M}\left\|Q_m A^{\alpha/2}\varphi_j(t)\right\|_2^2. \quad (3.17)$$

Combing (3.16) and (3.17) we have

$$\mathrm{Tr}F'\left((S(t)\,u_0)\circ q_M(t)\right) \leqslant -\frac{\nu}{2}\sum_{j=1}^{M}\left\|A^{1/2}\varphi_j(t)\right\|_2^2 -\frac{\mu}{2}\sum_{j=1}^{M}\left\|Q_m A^{\alpha/2}\varphi_j(t)\right\|_2^2+c_{\epsilon_p}\|\mathrm{Du}\|_q^q. \quad (3.18)$$

We will discard the term involving $Q_m A^{\alpha/2}\varphi_j$ in (3.18) to develop what we need for Theorems 1–3; it seems like a lot of potential power to throw away, but the alternative needs that M is significantly larger than m, which we want to avoid given the Kolmogorov intuition and the direct connection with our inertial-manifold estimates. For the term $\|Du\|_q^q$, we have by Holder's inequality that

$$\|Du\|_q^q \leqslant \|1\|_{q''}^q\,\|Du\|_2^q \leqslant |\Omega|^{\frac{2\alpha-3}{4\alpha}}\,\|Du\|_2^{1+3/(2\alpha)} \quad (3.19)$$

where we have used $q=1+3/(2\alpha)$ and $q''=(4\alpha+6)/(2\alpha-3)$. Now using that $\|Du\|_2 \leqslant \left\|A^{1/2}u\right\|_2$, using Holder's inequality on $\frac{1}{T}\int_0^T \left\|A^{1/2}u\right\|_2^q ds$, and using (3.11), (3.16), and (3.19), we have from (3.18) that to have $\limsup_{T\to\infty}\int_0^T \mathrm{Tr}F'\left((S(t))\,u_0\right)\circ q_m(t)\,dt \leqslant 0$ uniformly in space we need

$$\frac{\nu}{2}\sum_{j=1}^{M}\left\|A^{1/2}\varphi_j(t)\right\|_2^2 \geqslant \frac{c'_\alpha}{\mu^{3/(2\alpha)}}\lambda_m^{\frac{3(\alpha-1)}{2\alpha}}\,|\Omega|^{\frac{2\alpha-3}{4\alpha}}\left[\limsup_{T\to\infty}\frac{1}{T}\int_0^T\left\|A^{1/2}u\right\|_2^2 ds\right]^{\frac{2\alpha+3}{4\alpha}} \quad (3.20)$$

where

$$c'_\alpha = \frac{2\alpha}{3}\left[\frac{3}{2\alpha+3}\right]^{\frac{2\alpha+3}{2\alpha}} 2^{3/(2\alpha)} K_1^{3/(2\alpha)}. \tag{3.21}$$

Now, by ([33], Lemma VI.2.1)

$$\sum_{j=1}^{M} \left\| A^{1/2}\varphi_j(t)\right\|_2^2 \geqslant \lambda_1 + \ldots + \lambda_M \geqslant c'\lambda_1 M^{5/3} \tag{3.22}$$

since in 3D (see e.g. [33])

$$\lambda_j \sim c\lambda_1 j^{2/3}.$$

Here $c' = (3/5)c$ is a dimensionless constant depending only on the shape (and not the size) of Ω. Letting $c_\alpha = \left(2c'_\alpha\right)/c'$, we combine (3.22) with (3.20) to obtain the condition

$$M^{5/3} \geqslant \frac{c_\alpha}{\lambda_1 \nu \mu^{3/(2\alpha)}} \lambda_m^{\frac{3(\alpha-1)}{2\alpha}} [\Omega]^{\frac{2\alpha-3}{4\alpha}} \left[\limsup_{T\to\infty} \frac{1}{T}\int_0^T \left\|A^{1/2}u\right\|_2^2 ds\right]^{\frac{2\alpha+3}{4\alpha}} \tag{3.23}$$

to be satisfied for M. If we use (2.15) we obtain for $K'_\alpha = c_\alpha^{3/5}$ the estimate

$$M \geqslant \frac{K'_\alpha}{(\lambda_1\nu)^{3/5}\mu^{9/(10\alpha)}} \lambda_m^{\frac{9(\alpha-1)}{10\alpha}} |\Omega|^{\frac{6\alpha-9}{20\alpha}} \left[\frac{L_f^2}{\nu^2\lambda_1}\right]^{\frac{6\alpha+9}{20\alpha}}. \tag{3.24}$$

As in the introduction, set

$$\epsilon = \lambda_1^{3/2}\nu \limsup_{T\to\infty}\frac{1}{T}\int_0^T \left\|A^{1/2}u(s)\right\|_2^2 ds \tag{3.25}$$

then using $K'_\alpha = c_\alpha^{3/5}$ as above and (3.23) we have

$$M \geqslant \frac{K'_\alpha}{\lambda_1^{3/5}\nu^{3/5}} \frac{1}{\mu^{\frac{9}{10\alpha}}} \lambda_m^{\frac{9(\alpha-1)}{10\alpha}} |\Omega|^{\frac{6\alpha-9}{20\alpha}} \left[\frac{\epsilon}{\lambda_1^{3/2}\nu}\right]^{\frac{6\alpha+9}{20\alpha}}. \tag{3.26}$$

Now set

$$l_\epsilon = \left(\frac{\nu^3}{\epsilon}\right)^{1/4}; \tag{3.27}$$

l_ϵ is the Kolmogorov length scale; by rearranging terms in (3.26) we have for

$$l_0 = \frac{1}{\lambda_1^{1/2}} \tag{3.28}$$

that

$$M \geqslant \frac{K'_\alpha}{\nu^{3/5}\lambda_1^{3/5}\mu^{\frac{9}{10\alpha}}}\lambda_m^{\frac{9(\alpha-1)}{10\alpha}}\,|\Omega|^{\frac{6\alpha-9}{20\alpha}}\,\nu^{\frac{6\alpha+9}{10\alpha}}\,\lambda_1^{\frac{6\alpha+9}{40\alpha}}\left[\frac{l_0}{l_\epsilon}\right]^{\frac{6\alpha+9}{5\alpha}}. \tag{3.29}$$

Now note that

$$|\Omega| \sim c\lambda_1^{-3/2}; \tag{3.30}$$

setting $K_\alpha = c^{\frac{6\alpha-9}{20\alpha}}K'_\alpha$ and by combining and rearranging terms in (3.29) we have that

$$M \geqslant K_\alpha \left(\frac{\nu}{\mu}\right)^{\frac{9}{10\alpha}}\left[\lambda_m l_0^2\right]^{\frac{9(\alpha-1)}{10\alpha}}\left[\frac{l_0}{l_\epsilon}\right]^{\frac{6\alpha+9}{5\alpha}}. \tag{3.31}$$

We have used l_0 as is standard as a characteristic macroscopic length scale. This establishes (1.8b).

In the case $\mu \geqslant \nu$ it is still safe to assume $\lambda_m^{(\alpha-1)} \geqslant 1$; meanwhile we replace μ by ν in (3.16) and in the second, third, and fourth lines of (3.17); because of $\mu \geqslant \nu$ the last line of (3.17) remains the same. The replacement of μ by ν in (3.16) also means that we need to replace μ by ν in (3.23). Following through the development *without* combining with the other factors of ν allows us to see that we now need only replace μ by ν in (3.31). Thus we see that the condition $\mu \geqslant \nu$ allows us to eliminate the term $(\nu/\mu)^{\frac{9}{10\alpha}}$ which gives (1.8a). Thus Theorem 1 is established.

Meanwhile, note in (3.31) that $\lambda_m l_0^2 = \lambda_m/\lambda_1$ is independent of the size of Ω by virtue of $\lambda_m \sim c\lambda_1 m^{2/3}$ where c is dimensionless and depends only on the shape (but not the size) of Ω, and by design l_0/l_ϵ is similarly normalized; K_α depends only on α and K_1 from (3.9), and so depends only on the shape of Ω as well. Meanwhile ν is or can be taken to be dimensionless since typically it is chosen as the ratio of the mean free path and the characteristic macroscopic scale. Similar normalization considerations hold for μ. Thus all the bracketed terms in (3.31) are dimensionless, and may depend on the shape, but not the size, of Ω and are thus scale-invariant.

Now suppose that

$$\lambda_m \leqslant \left(\frac{1}{l_\epsilon}\right)^2 \tag{3.32}$$

then $\lambda_m l_0^2 \leqslant \left(\frac{l_0}{l_\epsilon}\right)^2$ and since

$$2\left(\frac{9\,(\alpha-1)}{10\alpha}\right)+\frac{6\alpha+9}{5\alpha}=\frac{15\alpha}{5\alpha}=3 \tag{3.33}$$

the condition (3.32) combined with (3.31) gives the simple condition

$$M \geqslant K_\alpha \left(\frac{\nu}{\mu}\right)^{\frac{9}{10\alpha}} \left(\frac{l_0}{l_\epsilon}\right)^3 \tag{3.34}$$

for $\mu \leqslant \nu$ and (1.10a) for $\mu \geqslant \nu$. With (3.24), (3.31), and (3.34) we have established (1.8) and (1.10a–1.10b). By substituting respectively $\mu \backsim \nu^\alpha$ and $\mu \sim c\nu$ we obtain (1.10c) and (1.10d).

For the rest of this section we discuss how (1.2) and the observations in [7] in the context of SEV motivate the choice $\mu \sim c\nu$ where $1/c$ is within α orders of magnitude of unity, and discuss an example of SVV methodology that uses $\mu \geqslant \nu$. The next technical results appear in Section 4.

To motivate the choice $\mu \sim c\nu$, we note that in [7] the spectral viscosity is studied in the limit $\delta \to l_\epsilon$, and since for at least theoretical reasons we want λ_m to be at least high enough to be in the neighborhood of l_ϵ, setting $k_\delta = l_\epsilon$ seems an appropriate starting point for letting (1.2) suggest a lower bound on μ. Letting $k \to k_\delta$ increases the size of (1.2) already by more than twofold over its value at $k=0$; as $k \to \infty$ the size of (1.2) increases by at least an order of magnitude.

A related quantity studied in [7] is $\nu_{\text{hyp}}\,(k, k_\delta)$, the spectral viscosity of a mixed hyperviscosity model for $\alpha = 2$. In ([7], Fig. 15) measurements of the ratio $\nu_{\text{hyp}}\,(k, k_\delta)/(\epsilon^{1/3}\delta^{4/3})$ are plotted against $k\delta$. Since in [7] $\epsilon^{1/3}\delta^{4/3}$ is used for viscosity, where ϵ is as in (3.25), ([7], Fig. 15) is basically a plot of μ/ν in this case (in the notation in [7] Δ denotes δ). A common value of the Kolmogorov constant is $C_{\text{k}} = 2.1$, for which the plot reflects most closely the profile (1.2). As k moves past $1/\delta \sim k_\delta$ the plot rises quickly in this case, and soon the ratio $\nu_{\text{hyp}}\,(k, k_\delta)/(\epsilon^{1/3}\delta^{4/3})$ is not orders-of-magnitude small but on the order of 15% or 20%. (Other values of C_{k} give different behavior, but still the ratio $\nu_{\text{hyp}}\,(k, k_\delta)/(\epsilon^{1/3}\delta^{4/3})$ stays between 5% and 15%.)

These considerations for the case $\alpha = 2$ suggest that it is reasonable to choose μ as large as $\mu \sim c\nu$ with $1/c$ within 2 orders of magnitude of unity when λ_m is at or beyond $1/l_\epsilon$, and in general within α orders of magnitude of unity for λ_m at or beyond $1/l_\epsilon$. To establish a number of our ideas and explore their theoretical consequences we have also looked at λ_m far beyond that, e.g. we have looked at $\lambda_m \leqslant (1/l_\epsilon)^2$. As noted, extrapolating in (1.2) for very large k suggests an increase of μ by at least an order of magnitude; extrapolating from the plot in ([7], Fig. 15) for $C_k = 2.1$ and very large k suggests an increase for μ up to $\mu \sim c\nu$ with $1/c$ within an order of magnitude of unity.

The 3D turbulent-channel simulations in [21] use $\mu \geqslant \nu$. Accurate agreement with both DNS simulations and experimental results are achieved in [21] for Reynolds numbers in the 100s, while the coefficient μ of the extra viscosity kernels is the reciprocal of the spectral or polynomial order P of the approximation, the largest value of P taken to be 21. The implication in SVV terms is that $\mu \to 0$ as $P \to \infty$, but the idea is that as $P \to \infty$ we approach direct numerical simulation of the NSE. Since the goal of SVV is to reduce the number of degrees of freedom needed for accurate simulation as compared with direct numerical simulation, P in general practice of SVV will generally be smaller than typical Reynolds numbers in turbulence simulations. This concludes our discussion of motivations for the choices $\mu \sim c\nu$ and $\mu \geqslant \nu$.

4. EXISTENCE OF AN INERTIAL MANIFOLD

For the results of this section we will use Theorem 3.2 of ([33], Chapter VIII), which we will refer to (and state below) as Theorem GFST. It generalizes the conditions of the main theorems of the Foias/Sell/Temam papers in a way that handles (1.4) for all $\alpha > 5/2$ in the case of general A_φ and for all $\alpha \geqslant 3/2$ for certain operators in the distinguished-class case. Theorem GFST applies to systems of the form

$$\frac{du}{dt} + A_1 u + R(u) = f, \tag{4.1a}$$

$$u(0) = u_0 \tag{4.1b}$$

for various general conditions on A_1, a linear operator with dense domain in a Hilbert space H, and R a bounded map from $D\left(A_1^\beta\right)$ into $D\left(A_1^{\beta-\gamma}\right)$ for β, γ non-negative constants to be determined below. For the system (1.4) we first set

$$A_1 = \mu A^\alpha \tag{4.2}$$

and

$$R(u) = \nu A - \mu P_m \left(A^\alpha - A_\varphi\right) - (u \cdot \nabla) u; \tag{4.3}$$

note that $\mu A_\varphi = \mu A^\alpha - \left(\mu P_m A^\alpha - \mu P_m A_\varphi\right)$. The first purpose of this section will be to show that (1.4) satisfies the conditions of Theorem GFST and therefore prove Theorem 5 above for A_1 and $R(u)$ as in (4.2), (4.3) and H as in the previous sections. Theorem GFST requires the following hypotheses:

(1) For every $u_0 \in D\left(A_1^{\beta}\right)$, (4.1) has a unique solution $u \in C\left(\mathbf{R}^+; D\left(A_1^{\beta}\right)\right) \cap L^2\left((0,T); D\left(A_1^{\beta}\right)\right) \forall T > 0$, and the mapping $S(t): u_0 \to u(t)$ is continuous from $D\left(A_1^{\beta}\right)$ into itself.

(2) $S(t)$ possesses an absorbing set $\mathcal{B}_0$ in $D\left(A_1^{\beta}\right)$, which is positively invariant $(S(t)\mathcal{B}_0 \subset \mathcal{B}_0 \forall t \geqslant 0)$. The ω-limit set of $\mathcal{B}_0$, denoted $\mathcal{A}$, is the maximal attractor for $S(\cdot)$ in $D\left(A_1^{\beta}\right)$.

(3) For some $\beta \geqslant 0$ and $\gamma \geqslant 0$ as in 5) below,

$$\left\| A_1^{\beta-\gamma} R(u) - A_1^{\beta-\gamma} R(v) \right\|_2 \leqslant C_M \left\| A_1^{\beta}(u-v) \right\|_2 \tag{4.4}$$

for all $u, v \in D\left(A^{\beta}\right)$, $\left\| A_1^{\beta} u \right\|_2 \leqslant M$, $\left\| A_1^{\beta} v \right\|_2 \leqslant M$.

(4) There exists a $\rho > 0$ such that the ball of radius $\rho/2$ centered at 0 in $D\left(A_1^{\beta}\right)$ is absorbing for (1.4).

(5) Let λ_N^1 be the eigenvalues of A_1, then there exists a function $K_{m_0} = K_{m_0}(N)$ such that for $N \geqslant m_0$

$$\lambda_{N+1}^1 - \lambda_N^1 \geqslant K_{m_0}(N)\left(\left(\lambda_{N+1}^1\right)^{\gamma} + \left(\lambda_N^1\right)^{\gamma}\right) \tag{4.5}$$

where $K_{m_0}(N) \to \infty$ as $N \to \infty$.

Note that the estimates in Section 2 give (1), (2), and (4). We will demonstrate (3) and (5) below.

Let P_N project onto the first N eigenspaces of A_1 and let $Q_N = I - P_N$. Let $\mathcal{F}_{b,l}^{\beta}$ be the class of Lipschitz functions Φ from $P_N D\left(A_1^{\beta}\right)$ into $Q_N D\left(A_1^{\beta}\right)$ satisfying

$$\mathrm{Supp}\Phi = \left\{ y \in P_N D\left(A_1^{\beta}\right) : \left\| A_1^{\beta} y \right\|_2 \leqslant 2\rho \right\}, \tag{4.6a}$$

$$\left\| A_1^{\beta} \Phi(y) \right\|_2 \leqslant b \ \forall y \in P_N D\left(A_1^{\beta}\right), \tag{4.6b}$$

$$\left\| A_1^{\beta} \Phi(y_1) - A_1^{\beta} \Phi(y_2) \right\|_2 \leqslant l \left\| A_1^{\beta}(y_1 - y_2) \right\|_2 \tag{4.6c}$$

where $b, l > 0$ are to be chosen.

The inertial manifold $\mathcal{M}$ will be the graph $\{y, \Phi(y)\}$ for $y \in P_N(H)$ of a fixed point Φ of a certain map $\mathcal{F}$; b, l are chosen in the proof of Theorem GFST so that the map $\mathcal{F}$ is a contraction on $\mathcal{F}^{\beta}_{b,l}$.

Now choose a C^{∞} function $\theta : \mathbf{R}^{+} \to [0,1]$ such that $\theta(s) = 1$ for $0 \leqslant s \leqslant 1, \theta(s) = 0$ for $s \geqslant 2$, and $\sup_{s \geqslant 0} |\theta'(s)| \leqslant 2$, and set $\theta_{\rho}(s) = \theta(s/\rho)$.

For

$$R_{\theta}(u) = \theta_{\rho}\left(\|A^{\alpha}u\|_2\right) R(u), \tag{4.7}$$

following [17, 18, 33] the "prepared equation" corresponding to (4.1) is

$$\frac{du}{dt} + A_1 u + R_{\theta}(u) = 0, \tag{4.8a}$$

$$u(0) = u_0. \tag{4.8b}$$

Now consider for $\Phi \in \mathcal{F}_{b,l}$ the two equations

$$\frac{dy}{dt} + A_1 y + P_N R_{\theta}(y + \Phi(y)) = P_N f, \tag{4.9a}$$

$$\frac{dz}{dt} + A_1 z + Q_N R_{\theta}(y + \Phi(y)) = Q_N f. \tag{4.9b}$$

The main ideas of the proof of Theorem GFST are as follows: let $y = y(t; y_0, \Phi)$ be the unique solution of (4.9a). Since $Q_N R_{\theta}(y + \Phi(y))$ is now a known function, we denote the solution of (4.9b) by $z = z(t; y_0, \Phi)$. The function $\mathcal{F}$ is defined as

$$(\mathcal{F}\Phi)(y_0) = z(0; y_0, \Phi)$$

and by uniqueness of (forward and backward) trajectories is well-defined as a map from $P_N D\left(A_1^{\beta}\right)$ to $Q_N D\left(A_1^{\beta}\right)$. In fact we have the formula

$$\mathcal{F}(\Phi(y_0)) = -\int_{-\infty}^{0} e^{\tau A_1} Q_N R_{\theta}(y(\tau) + \Phi(y(\tau)))\, d\tau. \tag{4.10}$$

With these considerations we can state

Theorem GFST *For a given $l \in (0, 1/8)$ there exists a $b > 0$ such that $\mathcal{F}$ is a strict contraction on $\mathcal{F}^{\beta}_{b,l}$, and therefore possesses a unique fixed point $\Phi_{\mathcal{M}} \in \mathcal{F}^{\beta}_{b,l}$. Furthermore, the graph of $\Phi_{\mathcal{M}}$ is an inertial manifold for (4.1).*

Theorem GFST applies to a wide variety of equations in mathematical physics, as detailed in [17, 18, 33]. We have shown in Section 2 that the hypothesis (1), (2), (4), are satisfied for (1.4), in particular we have an absorbing set $\{v : \|A^{\beta_0} v\|_2 \leqslant 2\rho\}$ if we take $\beta_0 = \beta/\alpha$ in (2.24) and take $\rho = 2\mu^{-\beta_0}\rho_\beta$ where ρ_β is defined immediately after (2.24). For the next part of this section we examine hypothesis (3) and (5) to show that Theorem GFST applies to (1.4), and thus prove Theorem 5.

For (3) we first verify that whenever $5 \leqslant 4\beta_1 + 4\gamma_1$

$$\left\| A^{\beta_1-\gamma_1} R_1(u) - A^{\beta_1-\gamma_1} R_1(v) \right\|_2 \leqslant C_M^1 \left\| A^{\beta_1}(u-v) \right\|_2 \tag{4.11}$$

for $A = -\Delta$ and

$$R_1(u) = (u \cdot \nabla)\, u. \tag{4.12}$$

Such estimates are basically shown in [17, 18, 33], at least in the special cases needed, but for completeness we present a complete development of (4.11) for all indicated values of β_1 and γ_1. We first assume that

$$s = \gamma_1 - \beta_1 \geqslant 0; \tag{4.13}$$

we have that

$$\begin{aligned} &\left\| A^{-s} R_1(u) - A^{-s} R_1(v) \right\|_2 \\ &\quad \leqslant \left\| A^{-s}\left((u-v) \cdot \nabla u\right) \right\|_2 + \left\| A^{-s}(v \cdot \nabla)(u-v) \right\|_2 . \end{aligned} \tag{4.14}$$

Now

$$\left\| A^{-s}(u-v) \cdot \nabla u \right\|_2 \leqslant M_2 \left\| (u-v) \cdot \nabla u \right\|_p \tag{4.15}$$

for $2 = 3p/(3-2sp)$ and where M_2 is M_1 for this choice of p, i.e. $p = 6/(3+4s)$. Then

$$\|(u-v)\,\nabla u\|_p \leqslant \|u-v\|_{ap}\, \|\nabla u\|_{bp} \tag{4.16}$$

where $1/a + 1/b = 1$. We want (note $\beta_1 = \gamma_1 - s$)

$$\|u-v\|_{ap} \leqslant M_3 \left\| A^{\beta_1}(u-v) \right\|_2 \tag{4.17}$$

where M_3 is the appropriate choice of M_1; (4.17) requires that

$$a = \frac{3+4s}{3+4s-4\gamma_1} \tag{4.18}$$

so that

$$b = \frac{3+4s}{4\gamma_1} \tag{4.19}$$

and

$$bp = \frac{3}{2\gamma_1}. \tag{4.20}$$

We then need

$$\|\nabla u\|_{bp} = \|\nabla u\|_{3/(2\gamma_1)} \leqslant M_4 \left\| A^{\beta_1} u \right\|_2; \tag{4.21}$$

after some arithmetic, this is seen to hold provided that

$$5 \leqslant 4\beta_1 + 4\gamma_1. \tag{4.22}$$

A similar development holds for the other term on the right-hand side of (4.14), for M_2, M_3, and M_4 as above. Let

$$M_5 = \max\{M_3, M_4\} \tag{4.23}$$

then for $\left\| A^{\beta_1} u \right\|_2 \leqslant M$, $\left\| A^{\beta_1} v \right\|_2 \leqslant M$, we have (4.11) for

$$C'_M = M_2 M_5 M. \tag{4.24}$$

We now show that (4.11) holds for

$$\beta_1 - \gamma_1 \geqslant 0. \tag{4.25}$$

The leading terms of $\left\| A^{\beta_1-\gamma_1} R_1 u \right\|_2$ are $\left\| A^{\beta_1-\gamma_1} u A^{1/2} u \right\|_2$ and $\left\| u A^{1/2+\beta_1-\gamma_1} u \right\|_2$. We have that

$$\left\| A^{\beta_1-\gamma_1} u A^{1/2} u \right\|_2 \leqslant \left\| A^{\beta_1-\gamma_1} u \right\|_{2a} \left\| A^{1/2} u \right\|_{2b} \tag{4.26}$$

where $1/a + 1/b = 1$. Choosing M_6 to be the appropriate value of M_1 we have that

$$\begin{aligned} \left\| A^{1/2} u \right\|_{2b} &\leqslant M_6 \left\| A^{\beta_1} u \right\|_2 \\ &= M_6 \left\| A^{\beta_1 - 1/2} A^{1/2} u \right\|_2 \end{aligned} \tag{4.27}$$

provided that $b = 3/(5 - 4\beta_1)$ and hence $a = 3/(4\beta - 2)$. Then there is an M_7 such that

$$\left\| A^{\beta_1-\gamma_1} u \right\|_{2a} \leqslant M_7 \left\| A^{\beta_1} u \right\|_2 \tag{4.28}$$

provided that $2a = 6/(4\beta_1 - 2) \leqslant 6/(3 - 4\gamma_1)$ which says that $4\beta_1 - 2 \geqslant 3 - 4\gamma_1$ or, again, $5 \leqslant 4\beta_1 + 4\gamma_1$ which is (4.22) for the case $\beta_1 - \gamma_1 \geqslant 0$.

It is now clear that similar inequalities hold when $A^{\beta_1-\gamma_1}$ is applied to $(u-v)\cdot\nabla u$ and $(v\cdot\nabla)(u-v)$. Thus, (4.11) holds whenever (4.22) is satisfied, with a C''_M replacing C'_M in (4.24), where M_6 and M_7 replace M_2 and M_5. We now establish (3) for $\alpha > 5/2$.

Replace A by μA^α in (4.11), then (4.4) holds with $\gamma=\gamma_1/\alpha$ and $\beta=\beta_1/\alpha$, and C''_M and C'_M now multiplied by $\mu^{-\gamma}$. There are two more terms in $R(u)$, namely $A_2=\mu P_m\left(A^\alpha - A_\varphi\right)$ and νA. In the sense of quadratic forms $A_1^{\beta-\gamma}\mu P_m\left(A^\alpha - A_\varphi\right) \leqslant A_1^\beta(\mu A^\alpha)^{-\gamma}\mu P_m A^\alpha = A_1^\beta \mu^{1-\gamma}P_m A^{\alpha-\gamma\alpha} \leqslant \mu^{1-\gamma}\lambda_m^{(1-\gamma)\alpha}A_1^\beta$ so that A_2 satisfies (4.4) by Poincaré with C_M replaced by $\mu^{1-\gamma}\lambda_m^{(1-\gamma)\alpha}$. For νA to satisfy (4.4), it is clear that we need $A^{\alpha(\beta-\gamma)}A \leqslant A^{\alpha\beta}$ or $A^{\alpha(\beta-\gamma+1/\alpha)} \leqslant A^{\alpha\beta}$, i.e.

$$\beta-\gamma+1/\alpha \leqslant \beta \tag{4.29}$$

or $1/\alpha \leqslant \gamma$, which says

$$1 \leqslant \gamma\alpha. \tag{4.30}$$

We will see below in handling (5) that α and γ satisfy

$$\gamma < \frac{2\alpha-3}{2\alpha} \tag{4.31}$$

in order to have a spectral gap; for (4.30) this means that

$$1 < \frac{2\alpha-3}{2} \tag{4.32}$$

which gives us the condition $\alpha > 5/2$. Thus, if (4.32) holds then νA satisfies (4.4) with $C'''_M = \nu/\epsilon$. Combining this with our observations for A_2 and $R_1(u)$, we have that (4.4) and hence (3) is satisfied for

$$C_M = \max\{\mu^{-\gamma}C'_M, \mu^{-\gamma}C''_M\} + \mu^{1-\gamma}\lambda_m^{(1-\gamma)\alpha} + \nu/\mu \tag{4.33}$$

provided that $\alpha > 5/2$ and

$$\frac{5}{\alpha} \leqslant 4\beta + 4\gamma, \tag{4.34}$$

i.e. $5 \leqslant 4\beta_1 + 4\gamma_1$. Note from (4.31) that we have

$$\frac{5}{\alpha} < 4\beta + \frac{4\alpha-6}{\alpha} \tag{4.35}$$

and we can take $\beta=0$ provided that $5 < 4\alpha - 6$ or

$$11/4 < \alpha. \tag{4.36}$$

For $5/2 < \alpha \leqslant 11/4$ we need $5 < 4\beta\alpha + 4\alpha - 6$ or $11 - 4\alpha < 4\beta_1$ or $11/4 - \alpha < \beta_1$; the supremum of the values needed for α in this range is $\beta_1 = 11/4 - 5/2 = 1/4$, easily handled by the bounds and absorbing set results on $A^{\beta_1} u$ established in Section 2. Thus (3) is satisfied for all $\alpha > 5/2$.

To establish (5) we use ([33], Lemma VIII.4.1) which states that

Lemma T. *If $\lambda_N^1 \sim cN^{\alpha_1}$ as $N \to \infty$ and $\alpha_1 > 1/(1-\gamma)$ then* (4.5) *is satisfied for arbitrarily large N's.*

Given that $\lambda_n \sim c\lambda_1 n^{2/3}$ we have that the eigenvalues of A_1 satisfy $\lambda_N^1 = \mu\lambda_N^\alpha \sim c\mu\lambda_1 N^{\alpha_1}$ for $\alpha_1 = (2/3)\alpha$. Solving for γ in the equation $(2/3)\alpha > 1/(1-\gamma)$ gives (4.31). Note that (4.31) is satisfied, i.e. gives a positive value of γ, whenever $\alpha > 3/2$, in particular when $\alpha > 5/2$. (Note that it is the development in (4.29–4.32) that dictates the condition $\alpha > 5/2$ rather than $\alpha > 3/2$.) With (1–5) satisfied for $\alpha > 5/2$, we thus have Theorem 5 by Theorem GFST. Note that the size of N depends on m from (4.33).

For Theorem 6 we let $\alpha \geqslant 3/2$ and consider A_φ such that $P_m A_\varphi = 0$ and $Q_m A_\varphi \geqslant Q_m A^\alpha$; we use a unique spectral-gap property of this operator, wherein by certain choices of A_1 we can produce a spectral gap between λ_{N+1}^1 and λ_N^1 with $N = m$ inherent in the structure of A_1 and independent of (4.31). We first prove Theorem 6 assuming $\alpha \geqslant 5/2$, $\mu \leqslant \nu$, and $A_\varphi = Q_m A^\alpha$ for which we can take $\gamma = 1/2$ and $\beta = 0$. The main ideas of the proof are simpler in this case and more closely resemble the arguments above. Now we set

$$A_1 = \mu P_m A + \mu A_\varphi = \mu (P_m A + Q_m A^\alpha), \tag{4.37}$$

$$R(u) = (u \cdot \nabla) u + (\nu - \mu) P_m A + \nu Q_m A. \tag{4.38}$$

Borrowing a technique from [1] we set $A_1 = \mu N^{\alpha-1} A^\alpha$ where $N = P_m A^{-1} + Q_m I$. Note that for $s > 0$

$$N^{-s} \leqslant \lambda_m^s I \tag{4.39}$$

in the sense of quadratic forms. Thus all the above estimates involving (4.11) and (4.34) hold for the new A_1 in place of μA^α with the modification from (4.39) that with $s = \gamma - \beta = \gamma = 1/2$ we need to multiply C'_M and C''_M above by $\mu^{-1/2}\lambda_m^{(\alpha-1)/2}$. To obtain (4.4) it remains to take care of the term $A_2 \equiv (\nu - \mu) P_m A + \nu Q_m A$ in $R(u)$: note that $(\nu - \mu) A \leqslant A_2 \leqslant \nu A$ in the sense of quadratic forms, so since $\alpha \geqslant 5/2 > 2$ we have from (4.39) that

$A_1^{-1/2}A_2 \leqslant \nu A_1^{-1/2}A = \nu\mu^{-1/2}N^{(\alpha-1)/2}A^{-\alpha/2}A \leqslant \nu\mu^{-1/2}\lambda_m^{(\alpha-1)/2}A^{-((\alpha/2)-1)} \leqslant \nu\mu^{-1/2}\lambda_m^{(\alpha-1)/2}\lambda_1^{-((\alpha/2)-1)}I$. Thus we now obtain (3) with

$$C_M = \mu^{-1/2}\lambda_m^{(\alpha-1)/2}\max\{C'_M, C''_M\} + \nu\mu^{-1/2}(\lambda_m^{(\alpha-1)/2}\lambda_1^{-((\alpha/2)-1)}). \quad (4.40)$$

Now (2.24) reduces to (2.13) since $\beta = 0$ and thus we can take for some $\delta > 0$

$$\rho = 2(1+\delta)\left(\frac{L_f}{\nu\lambda_1}\right). \quad (4.41)$$

Note (1–4) are now satisfied for the new choices of A_1 and $R(u)$ in (4.37), (4.38).

Now set

$$M_1^T = \sup_{\left\|A_1^{\beta_1}u\right\| \leqslant 2\rho} \left\|A_1^{\beta_1-\gamma_1}R(u)\right\|_2; \quad (4.42a)$$

note that since $R(0) = 0$

$$M_1^T \leqslant C_{2\rho} \quad (4.42b)$$

where $C_{2\rho}$ is C_M with $M = 2\rho$; set

$$M_2^T = \frac{2M_1^T}{\rho} + C_{2\rho}. \quad (4.42c)$$

As in the proof of Theorem GFST, for $\sigma \geqslant 0$ set

$$\kappa_2(\sigma) = \sigma^\sigma e^{-\sigma}, \kappa_3(\sigma) = e^{-\sigma} + \frac{\kappa_2(\sigma)^{1-\sigma}}{1-\sigma}, \quad (4.43)$$

and

$$\kappa_4 = \kappa_3(\gamma) = \kappa_3(1/2). \quad (4.44)$$

Given $l \in (0, 1/8)$ the conditions on which the dimension of $\mathcal{M}$ depends in Theorem GFST are that

$$\lambda_{N+1}^1 > (M_2^T)^2\left\{\frac{1+l}{l} + 4\kappa_4 + 11\right\}^{1/2} \quad (4.45)$$

and

$$\lambda_{N+1}^1 - \lambda_N^1 \geqslant 2M_2^T\left(\frac{1+l}{l}\right)\left(\left(\lambda_{N+1}^1\right)^{1/2} + \left(\lambda_N^1\right)^{1/2}\right). \quad (4.46)$$

We want to show explicity that (4.45) and (4.46) are satisfied. Recall that we are taking $N=m$. We first note that since $\lambda^1_{N+1}=\lambda^1_{m+1}=\mu\lambda^\alpha_{m+1}$ and $\lambda^1_N=\lambda^1_m=\mu\lambda_m$ we have that

$$\begin{aligned}\lambda^1_{N+1}-\lambda^1_N &= \mu\lambda^\alpha_{m+1}-\mu\lambda_m)\\ &= \mu^{1/2}\left(\lambda^{\alpha/2}_{m+1}-\lambda^{1/2}_m\right)\left((\lambda^1_{N+1})^{1/2}+(\lambda^1_N)^{1/2}\right).\end{aligned} \tag{4.47}$$

Using $\lambda_m\sim cm^{2/3}$ we have

$$\begin{aligned}\lambda^{\alpha/2}_{m+1}-\lambda^{1/2}_m &\sim (c\lambda_1)^{1/2}\left[(c\lambda_1)^{(\alpha-1)/2}(m+1)^{\alpha/3}-m^{1/3}\right]\\ &\geqslant (c\lambda_1)^{1/2}\left[(c\lambda_1)^{(\alpha-1)/2}m^{\alpha/3}-m^{1/3}\right]\\ &= (c\lambda_1)^{1/2}m^{1/3}\left[(c\lambda_1)^{(\alpha-1)/2}m^{(\alpha-1)/3}-1\right]\\ &\geqslant (c\lambda_1)^{1/2}m^{1/3}\left[(c\lambda_1)^{(\alpha-1)/2}(m^{(\alpha-1)/3}-(1/2)m^{(\alpha-1)/3})\right]\\ &= (1/2)\,(c\lambda_1)^{1/3}m^{1/3}\left[(c\lambda_1)^{(\alpha-1)/2}m^{(\alpha-1)/3}\right]\\ &= (1/2)\,(c\lambda_1)^{\alpha/2}m^{\alpha/3},\end{aligned} \tag{4.48}$$

where it is safe to assume that m is large enough so that $(c\lambda_1)^{(\alpha-1)/2}(1/2)m^{(\alpha-1)/3}\geqslant 1$. With (4.48) we have (5).

To satisfy (4.45) using $\lambda^1_{m+1}=\mu\lambda^\alpha_{m+1}\sim\mu c^\alpha\lambda^\alpha_1 m^{(2\alpha)/3}$ we want

$$m>\mu^{-3/(2\alpha)}(c\lambda_1)^{-3/2}(M^T_2)^{3/\alpha}\left\{\frac{1+l}{l}+4\kappa_4+11\right\}^{3/(4\alpha)}. \tag{4.49}$$

To satisfy (4.46), from (4.47) and (4.48) we want

$$m>2^{3/\alpha}\mu^{-3/(2\alpha)}(c\lambda_1)^{-3/2}(M^T_2)^{3/\alpha}\left[(1+l)\,/\,l\right]^{3/\alpha}. \tag{4.50}$$

Meanwhile, there is dependence on m in M^T_2 coming from $C_{2\rho}$; recall that $C_{2\rho}$ is defined from (4.41) by replacing M with 2ρ. Since $C'_M=M_2M_5M$ and $C''_M=M_6M_7M$, if we set

$$C_0=\max\{2M_2M_5,\,2M_6M_7\} \tag{4.51}$$

then we have from (4.40) and (4.51) that

$$\begin{aligned}C_{2\rho} &= 2\mu^{-1/2}\lambda^{(\alpha-1)/2}_m C_0\rho+\nu\mu^{-(1/2)}\left(\lambda^{(\alpha-1)/2}_m\lambda_1^{-((\alpha/2)-1)}\right)\\ &= \lambda^{(\alpha-1)/2}_m\left[2\mu^{-1/2}C_0\rho+\nu\mu^{-1/2}\lambda_1^{-((\alpha/2)-1)}\right]\\ &\sim (c\lambda_1)^{(\alpha-1)/2}m^{(\alpha-1)/3}\left[2\mu^{-1/2}C_0\rho+\nu\mu^{-1/2}\lambda_1^{-((\alpha/2)-1)}\right]\\ &\equiv m^{(\alpha-1)/3}C^{\mu,\nu,\rho}_1.\end{aligned} \tag{4.52}$$

Now from (4.42b), (4.42c) and (4.52) we have that

$$(M_2^T)^{3/\alpha} \leqslant [(2C_{2\rho})/\rho + C_{2\rho}]^{3/\alpha} \leqslant m^{(\alpha-1)/\alpha}[(2C_1^{\mu,\nu,\rho})/\rho + C_1^{\mu,\nu,\rho}]^{3/\alpha}$$
$$\equiv m^{(\alpha-1)/\alpha}(C_2^{\mu,\nu,\rho})^{3/\alpha}, \qquad (4.53)$$

thus from (4.49) we have that (4.45) is satisfied if

$$m > \mu^{-3/(2\alpha)}(c\lambda_1)^{-3/2} m^{(\alpha-1)/\alpha}(C_2^{\mu,\nu,\rho})^{3/\alpha}\left\{\frac{1+l}{l} + 4\kappa_4 + 11\right\}^{3/(4\alpha)}, \quad (4.54)$$

and from (4.50) we have that (4.46) is satisfied if

$$m > 2^{3/\alpha}\mu^{-3/(2\alpha)}(c\lambda_1)^{-3/2} m^{(\alpha-1)/\alpha}(C_2^{\mu,\nu,\rho})^{3/\alpha}[(1+l)/l]^{3/\alpha}, \qquad (4.55)$$

from which by solving for m we obtain the conditions

$$m > \mu^{-3/2}(c\lambda_1)^{-(3\alpha)/2}(C_2^{\mu,\nu,\rho})^3[(1+l)/l + 4\kappa_4 + 11]^{3/4}, \qquad (4.56)$$

and

$$m > 8\mu^{-3/2}(c\lambda_1)^{-(3\alpha)/2}(C_2^{\mu,\nu,\rho})^3[(1+l)/l]^3, \qquad (4.57)$$

for m to satisfy. Thus we have an inertial manifold if m is larger than the maximum of the right-hand sides of (4.56) and (4.57). Reversing $\lambda_m \sim cm^{2/3}$ we have

$$\lambda_m > \mu^{-1}c^{1-\alpha}\lambda_1^{-\alpha}(C_2^{\mu,\nu,\rho})^2[(1+l)/l + 4\kappa_4 + 11]^{1/2}, \qquad (4.58)$$

and

$$\lambda_m > 4\mu^{-1}c^{1-\alpha}\lambda_1^{-\alpha}(C_2^{\mu,\nu,\rho})^2[(1+l)/l]^2, \qquad (4.59)$$

for λ_m to satisfy; we will refine the estimates (4.58), (4.59) below. This established Theorem 6 in the special case $\alpha \geqslant 5/2$, $\mu \leqslant \nu$, and $A_\varphi = Q_m A^\alpha$.

We now prove Theorem 6 for all values of $\alpha \geqslant 3/2$, all A_φ satisfying $P_m A_\varphi = 0$ and $Q_m A_\varphi \geqslant Q_m A^\alpha$, and all positive μ, ν. The spectral gap between λ^1_{m+1} and λ^1_m inherent in the structure of a similar A_1 will still work best with the choice $\gamma = 1/2$ which by (4.34) means that we need $\beta \geqslant 5/(4\alpha) - 1/2$ in the case $3/2 \leqslant \alpha < 5/2$; for $\alpha \geqslant 5/2$ we can still take $\beta = 0$. Given these choices of γ and β we will again develop conditions that will show how big m must be to guarantee the existence of an inertial manifold. Let $c_1 = \min\{\mu, \nu\}$ and let $c_2 = \max\{\nu - \mu, \mu - \nu\}$; set $A_2 = c_1(P_m A + Q_m A_\varphi)$, and set $A_3 = P_m A$ for $\mu \leqslant \nu$ and $A_3 = Q_m A_\varphi$ for $\mu \geqslant \nu$. Then we now take

$$A_1 = \nu A + \mu A_\varphi = c_1(P_m A + Q_m A_\varphi) + c_2 A_3 + \nu Q_m A$$
$$\equiv A_2 + c_2 A_3 + \nu Q_m A, \qquad (4.60)$$

which allows us to simply take

$$R(u) = (u \cdot \nabla) u. \tag{4.61}$$

We set $A_4 = c_1(P_m A + Q_m A^\alpha) = c_1 N^{\alpha-1} A^\alpha$ where $N = P_m A^{-1} + Q_m$ as above. Working with (4.11), (4.34), and (4.39) as before, we can calculate that A_4 satisfies (4.4) with $C_M = (1/c_1)^\eta \lambda_m^{\eta(\alpha-1)} \max\{C'_M, C''_M\}$; note again that we are using $\gamma = 1/2$ and $\eta \equiv \gamma - \beta = 1/2 - \beta$, with $\beta = 5/(4\alpha) - 1/2$ for $3/2 \leqslant \alpha < 5/2$ and $\beta = 0$ for $\alpha \geqslant 5/2$. Now by the functional calculus $A_1^{-s} \leqslant A_2^{-s} \leqslant A_4^{-s}$ while $A_4^s \leqslant A_2^s \leqslant A_1^s$ so that $\left\| A_1^{\beta-\gamma}(R(u) - R(v)) \right\|_2 \leqslant \left\| A_4^{\beta-\gamma}(R(u) - R(v)) \right\|_2 \leqslant C_M \left\| A_4^\beta (u - v) \right\|_2 \leqslant C_M \left\| A_1^\beta (u - v) \right\|_2$. This means that A_1 also satisfies (4.4) with C_M, β, γ, and η as above. We now have (3) with

$$C_M = c_1^{-\eta} \lambda_m^{\eta(\alpha-1)} C_0 M \tag{4.62}$$

where C_0 is as in (4.51).

For the case $7/4 < \alpha < 5/2$, set $c_0 = \max\{\mu, \nu\}$, then since $A_1^\beta \leqslant (c_0 A^\alpha)^\beta$ we can take for some $\delta > 0$

$$\rho = c_0^{-\beta}(1+\delta)(\rho_{m,\beta_1})^{1/2} \tag{4.63}$$

where ρ_{m,β_1} is defined in (2.24) with $\beta_1 = \alpha\beta$ replacing β. For $3/2 \leqslant \alpha \leqslant 7/4$ we use the "messier" estimates that can be bootstrapped from (2.22) mentioned in Section 2. For $\alpha \geqslant 5/2$ we again take ρ as in (4.41). We now have that (1–4) are satisfied for the new choices of A_1 and $R(u)$ in (4.37), (4.38). In particular, for ρ as in (4.41) or (4.63) we thus have that

$$\begin{aligned} C_{2\rho} &= 2(c_1)^{-\eta} \lambda_m^{\eta(\alpha-1)} C_0 \rho \\ &\sim (cm^{2/3})^{\eta(\alpha-1)} [2(c_1)^{-\eta} C_0 \rho] \\ &= 2(c\lambda_1)^{\eta(\alpha-1)} m^{2\eta(\alpha-1)/3} [(c_1)^{-\eta} C_0 \rho], \end{aligned} \tag{4.64}$$

and, combining (4.42b), (4.42c), and (4.64) we thus have that

$$M_2^T \leqslant (2C_{2\rho})/\rho + C_{2\rho} = m^{2\eta(\alpha-1)/3} [2(c\lambda_1)^{\eta(\alpha-1)} c_1{}^{-\eta} C_0 (2+\rho)]. \tag{4.65}$$

Meanwhile, since $c_2 A_3$ and $\nu Q_m A$ are positive operators, since $A_4 \leqslant A_2 \leqslant A_1$, and since $\lambda_{m+1}^1 \geqslant \mu \lambda_{m+1}^\alpha$, we have that (4.47) becomes $\lambda_{N+1}^1 - \lambda_N^1 = \left((\lambda_{m+1}^1)^{1/2} - (\lambda_m^1)^{1/2}\right)\left((\lambda_{m+1}^1)^{1/2} + (\lambda_m^1)^{1/2}\right) \geqslant ((\mu\lambda_{m+1}^\alpha)^{1/2} - (\nu\lambda_m)^{1/2}) \left((\lambda_{m+1}^1)^{1/2} + (\lambda_m^1)^{1/2}\right)$, while (4.48) becomes

$$\mu^{1/2}\lambda_{m+1}^{\alpha/2} - \nu^{1/2}\lambda_m^{1/2} \sim (c\lambda_1)^{1/2}\left[(c\lambda_1)^{(\alpha-1)/2}\mu^{1/2}(m+1)^{\alpha/3} - \nu^{1/2}m^{1/3}\right]$$
$$\geqslant (c\lambda_1)^{1/2}\left[(c\lambda_1)^{(\alpha-1)/2}\mu^{1/2}m^{\alpha/3} - \nu^{1/2}m^{1/3}\right]$$
$$= (c\lambda_1)^{1/2}m^{1/3}\left[(c\lambda_1)^{(\alpha-1)/2}\mu^{1/2}m^{(\alpha-1)/3} - \nu^{1/2}\right]$$
$$\geqslant (c\lambda_1)^{1/2}m^{1/3}\left[(c\lambda_1)^{(\alpha-1)/2}\mu^{1/2}(m^{(\alpha-1)/3} - (1/2)m^{(\alpha-1)/3})\right]$$
$$= (1/2)\mu^{1/2}(c\lambda_1)^{1/2}m^{1/3}\left[(c\lambda_1)^{(\alpha-1)/2}m^{(\alpha-1)/3}\right]$$
$$= (1/2)\mu^{1/2}(c\lambda_1)^{\alpha/2}m^{\alpha/3}, \tag{4.66}$$

where we have assumed that $(1/2)(c\lambda_1)^{(\alpha-1)/2}\mu^{1/2}m^{(\alpha-1)/3} \geqslant \nu^{1/2}$ or $m^{(\alpha-1)/3} \geqslant 2(c\lambda_1)^{-(\alpha-1)/2}(\nu/\mu)^{1/2}$; with this assumption, which we will need to remember later, we now have (5). Combining (4.42a–4.42c) with (4.64), (465) and $\lambda_{m+1}^1 \geqslant \mu\lambda_{m+1}^{\alpha}$, and setting $p_\alpha \equiv [\alpha - 2\eta(\alpha-1)]/3$, we have that (4.45) is satisfied if

$$m^{p_\alpha} > 2\mu^{-1/2}(c\lambda_1)^{\eta(\alpha-1)-\alpha/2}(1/c_1)^{\eta}C_0(2+\rho)]\left\{\frac{1+l}{l} + 4\kappa_4 + 11\right\}^{1/4}, \tag{4.67}$$

where we have taken the square-root of both sides of (4.45) after plugging in. Combining (4.42a–4.42c) with (4.64), (465) and (4.66) we have that (4.46) is satisfied if

$$m^{p_\alpha} > 8\mu^{-1/2}(c\lambda_1)^{\eta(\alpha-1)-\alpha/2}(1/c_1)^{\eta}C_0(2+\rho)][(1+l)/l]. \tag{4.68}$$

Thus if m^{p_α} is larger than the maximum of the right-hand sides of (4.67) and (4.68), *and if* $m^{(\alpha-1)/3} \geqslant 2(c\lambda_1)^{-(\alpha-1)/2}(\nu/\mu)^{1/2}$ which gave (4.66) and thus (5), we have an inertial manifold $\mathcal{M}$ as before. This finishes the proof of Theorem 6; though the proof is somewhat more involved than the proof we used for $\alpha \geqslant 5/2$, the estimates (4.67) and (4.68) in fact refine those in (4.58), (4.59) as we will see below.

We now explore what conditions on m are implied by (4.68), (4.69). Recall that $\eta \equiv \gamma - \beta = 1 - 5/(4\alpha)$ if $3/2 \leqslant \alpha < 5/2$. Thus we have, for example, that $p_\alpha = 4/9$ if $\alpha = 3/2$, and that $p_\alpha = 5/12$ if $\alpha = 2$. For $\alpha \geqslant 5/2$ we take $\beta = 0$ so that $\eta = 1/2$, $\eta(\alpha-1) - \alpha/2 = -(1/2)$, and $p_\alpha = 1/3$, in which case the estimates (4.67), (4.68) simplify to

$$m > \mu^{-3/2}(c\lambda_1)^{-3/2}[2(1/c_1)^{1/2}C_0(2+\rho)]^3[(1+l)/l + 4\kappa_4 + 11]^{3/4}; \tag{4.69}$$

and

$$m > \mu^{-3/2}(c\lambda_1)^{-3/2}[8(1/c_1)^{1/2}C_0(2+\rho)]^3[(1+l)/l]^3 \tag{4.70}$$

for m to satisfy. Reversing $\lambda_m \sim c\lambda_1 m^{2/3}$ in this case and simplifying we have

$$\lambda_m > 4(\mu c_1)^{-1}[C_0(2+\rho)]^2[(1+l)/l + 4\kappa_4 + 11]^{1/2}, \tag{4.71}$$

and

$$\lambda_m > 64(\mu c_1)^{-1}[C_0(2+\rho)]^2[(1+l)/l]^2, \tag{4.72}$$

for λ_m to satisfy, along with $m^{(\alpha-1)/3} \geqslant 2(c\lambda_1)^{-(\alpha-1)/2}(\nu/\mu)^{1/2}$ which becomes $\lambda_m \geqslant [2(\nu/\mu)]^{1/(\alpha-1)}$. For $\alpha \geqslant 5/2$ the latter requires that $\lambda_m \geqslant [2(\nu/\mu)]^{2/3}$, which is less stringent an assumption than the requirements of (4.71) and (4.72). Note that (4.71) and (4.72) are cleaner estimates in terms of the key parameters μ, c_1, and ρ than (4.58) and (4.59).

We can estimate how large λ_m has to be in (4.71), (4.72) by using (4.41) and (4.63); in the case $\alpha \geqslant 5/2$ where we take $\beta = 0$ we can from (4.41) set $\rho = L_f/(\nu\lambda_1) = \nu\lambda_1^{1/4}G$. The right-hand sides of (4.71), (472) in this case are on the order of generic constants times $(\mu c_1)^{-1}(2+\nu\lambda_1^{1/4}G)^2$. This is larger than our estimates on the attractor as discussed above when expressed in terms of G, both in terms of a higher power on G and because of the extra term $(\mu c_1)^{-1}$. On the other hand, these estimates are not wildly larger than our attractor estimates, so we can be somewhat satisfied with them, especially given the fact that $\mathcal{A} \subset \mathcal{M}$. When $\alpha < 5/2$ we can expect significantly larger estimates, given that now we need to take ρ as in (4.63) or bootstrapped from that as discussed above. Meanwhile the case $\alpha \geqslant 5/2$ leaves out the case $\alpha = 2$ often used in practice, but does include the case $\alpha = 3$ used in [10] as well as the higher values used in [5, 6] as noted.

For Theorem 7, since now A_φ satisfies the conditions of Theorem 6 for m_0 playing the role of m and μ_{m_0+1} playing the role of μ, we simply make these replacements throughout the proof of Theorem 6. Since we expect μ_{m_0+1} to be significantly smaller than μ, the estimates become significantly larger, in particular we can expect that $c_1 = \mu_1$.

For A_φ' as in Theorem 8, we show that A_φ satisfies the conditions of Theorem 6 for m_1 playing the role of m and μ_{m_1+1} playing the role of μ. After making these replacements, and for simplicity setting $m_1 = m$ and $\eta_{m_1} = \eta$, the only significant difference in the proof is that (4.66) needs to be replaced by:

$$\begin{aligned}
\mu\lambda_{m+1}^{\alpha/2} - \nu\lambda_m^{(\eta\alpha)/2} &\sim (c\lambda_1)^{(\eta\alpha)/2}\left[(c\lambda_1)^{[(1-\eta)\alpha]/2}\,\mu\,(m+1)^{\alpha/3} - \nu m^{(\eta\alpha)/3}\right]\\
&\geqslant (c\lambda_1)^{(\eta\alpha)/2}\left[(c\lambda_1)^{[(1-\eta)\alpha]/2}\,\mu m^{\alpha/3} - \nu m^{(\eta\alpha)/3}\right]\\
&= (c\lambda_1)^{(\eta\alpha)/2}\, m^{(\eta\alpha)/3}\left[(c\lambda_1)^{[(1-\eta)\alpha]/2}\,\mu m^{[(1-\eta)\alpha]/3} - \nu\right]\\
&\geqslant (c\lambda_1)^{(\eta\alpha)/2}\, m^{(\eta\alpha)/3}\Big[(c\lambda_1)^{[(1-\eta)\alpha]/2}\,(m^{[(1-\eta)\alpha]/3}\\
&\quad - (1/2)m^{[(1-\eta)\alpha]/3})\Big]\\
&= (1/2)\,(c\lambda_1)^{(\eta\alpha)/2}\, m^{(\eta\alpha)/3}\left[(c\lambda_1)^{[(1-\eta)\alpha]/2}\, m^{[(1-\eta)\alpha]/3}\right]\\
&= (1/2)\,(c\lambda_1)^{\alpha/2}\, m^{\alpha/3}, \qquad (4.73)
\end{aligned}$$

where we are assuming that m is large enough that $(c\lambda_1)^{[(1-\eta)\alpha]/2}(1/2)$ $\mu m^{[(1-\eta)\alpha]/3} \geqslant \nu$; this should not significantly increase our estimates unless η is very close to 1. This concludes the proof of Theorem 8.

5. CONCLUSION

We imagine a variety of generalizations of Theorems 6–8 may be possible given further exploration of the class of operators A_φ. Such generalizations could include in particular versions of the perturbation results in [17, 18, 33] adapted to these settings.

The results in [15] are among a number of important results obtained for the closure model variously known as the LANS-α model, the 3D Camassa–Holm equations, or simply NS-α model. This model was derived in its inviscid form in [19, 20] with the goal of providing a closure model for incompressible flow in which all of the geometrical, and in particular invariant, properties of the inviscid dynamics are retained. In the viscous form, the equations take the standard NSE ((1.1) with $S_{\text{sg}} = 0$) and replace u_t with $(\alpha_0^2 I + \alpha_1^2 A)u_t$ and $u \times (\nabla \times u)$ with $u \times (\nabla \times (\alpha_0^2 I + \alpha_1^2 Au))$.

Global regularity of these equations on a periodic box and subsequence convergence to Leray solutions of the NSE as $\alpha_1 \to 0$ (and $\alpha_0 \to 1$) is established in [15], as well as estimates on the Hausdorff and fractal dimension of the attractor. In particular it is shown in [15] that $\dim_{\text{H}} \mathcal{A} \leqslant \dim_{\text{F}} \mathcal{A} \leqslant c(\alpha_1^2\lambda_1)^{-3/4}[l_0/l_\epsilon]^3 \sim c\alpha_1^{-3/2}|\Omega|^{1/2}[l_0/l_\epsilon]^3$ for a generic constant c. The power on l_0/l_ϵ matches the Landau–Lifschitz prediction and also matches the estimates on invariant sets bounded in $V = PH^1(\Omega)$ for weak solutions of the 3D NSE as shown in [11–13]. There is also no potential to "absorb" the growth term $(1/\alpha_1)^{3/2}$, so the estimate simply grows without bound as $\alpha_1 \to 0$. In contrast the estimates (1.8), (1.10) have significant leeway to allow for values of m large enough so that λ_m is past the inertial range, suggesting good to very good NSE approximation and very good agreement with the Landau–Lifschitz theory as discussed following (1.10). The estimates (1.8), (1.10) are also scale-invariant.

The NS-α model has interesting physical properties. For further references on results for the NS-α model as well as mathematical and physical properties in both the viscous and inviscid cases, see the references and discussion in e.g. [1, 15, 28]. In [1] we obtained global regularity and subsequence-convergence results for a spectrally-implemented version of the NS-α that are the analogues of those we obtained therein for (1.4), and in a future paper we plan to develop attractor estimates for this spectral NS-α model along the lines of the estimates developed here.

The Kolmogorov theory predicts that the inertial range, i.e. wavenumber modes corresponding to wavenumbers below $k_{\mathrm{d}}=1/l_\epsilon$, behaves almost inviscidly, while the wavenumber modes above the inertial range are quickly dissipated by viscosity. The idea of (1.4), like the SEV and SVV models which motivate it, is to enforce suppression of high wavenumbers while preserving the inertial range. If the inertial range sees only standard NSE viscosity, this is virtually assured, since the Kolmogorov theory predicts that the wavenumber modes in the dissipative range quickly become of no dynamical consequence.

It makes sense particularly for high Reynolds numbers to keep the inertial range free of extra regularization because the actual convective process needs some time to cascade energy to the dissipative scales; before then some of the energy transferred from larger to smaller scales in the inertial range will transfer *back*, or "backscatter" to the larger scales. Indeed, "the convective nature of cascades is reflected in the presence of the energy backscatter" [6]. This reinforces the motivation to keep all of the dynamical structures of the convective cascade in the inertial range intact.

These observations together with the Kolmogorov theory, in which "viscosity provides an ultraviolet cutoff at a dissipation wavenumber k_{d}"[6], suggest the suitability of a multiscale model. This is reflected throughout the development here, but particularly in Theorems 6–8, which directly use the multiscale aspects of the model to produce a wider spectral gap than otherwise.

Estimates on the number of degrees of freedom for the NSE and its closure models are a measure of the complexity of the system. In addition to upper bounds, lower bounds on the dimension of the attractor for the 2D NSE have been obtained as well; see the discussion and references in [16] and [33]. Another interesting way to obtain a lower-bound estimate on the complexity is to provide upper bounds on the size of the nodal set for the vorticity, as was done in [25, 26] for periodic solutions of the 2D

NSE. It would be interesting to obtain estimates for (1.4) in this context in 3D and to see how the estimates depend on m.

REFERENCES

1. Avrin, J. Approximate globally-regular solutions of the Navier–Stokes equations via spectral methods, submitted.
2. Avrin, J. (1996). Large-eigenvalue global existence and regularity results for the Navier–Stokes equation. *J. Diff. Eqns.* **12**, 365–390.
3. Avrin, J. (2003). Singular initial data and uniform global bounds for the hyperviscous Navier–Stokes equations with periodic boundary conditions. *J. Diff. Eq.* **190**, 330–351.
4. Ball, J. (1997). Continuity properties and global attractors of generalized semiflows and the Navier–Stokes equations. *J. Nonlinear Sci.* **7**, 475–502. Erratum: *J. Nonlinear Sci.* **8** (1998).
5. Borue, V., and Orszag, S. (1996). Numerical study of three-dimensional Kolmogorov flow at high Reynolds numbers. *J. Fluid Mech.* **306**, 293–323.
6. Borue, V., and Orszag, S. (1998). Local energy flux and subgrid-scale statistics in three-dimensional turbulence. *J. Fluid Mech.* **306**, 1–31.
7. Cerutti, S., Meneveau, C., and Knio, O. M. (2000). Spectral and hyper-eddy viscosity in high-Reynolds-number turbulence. *J. Fluid Mech.* **421**, 307–338.
8. Chen, G. Q., Du, Q., and Tadmor, E. (1993). Spectral viscosity approximations to multi-dimensional conservation laws. *Math. Comp.* **61**, 629.
9. Chollet, J. (1985). Two point closure used for a sub-grid scale model in LES. In Bradbury, J. L., Durst, F., Launder, B., Schmidt, F., and Whitelaw, J. (eds.), *Turbulent Shear Flows*, Vol. 4, Springer, pp. 62–72.
10. Chollet, J., and Lesieur, M. (1981). Parametrization of small scales of three-dimensional isotropic turbulence utilizing spectral closures. *J. Atmos. Sci.* **38**, 2747–2757.
11. Constantin, P., and Foias, C. F. (1985). Global Liapunov exponents, Kaplan-Yorke formulas and the dimension of the attractor for the 2-d Navier–Stokes equations. *Com. Pure Appl. Math* **38**, 1–27.
12. Constantin, P., Foias, C., Manley, O., and Temam, R. (1985). Determining modes and fractral dimensions of turbulent flows. *J. Fluid Mech.* **150**, 427–440.
13. Constantin, P., Foias, C. F., and Temam, R. (1985). Attractors representing turbulent flows. *Mem. Am. Math. Soc.* **53**, 114.
14. Constantin, P., Foias, C., and Temam, R. (1988). On the dimension of the attractors in two dimensional turbulence. *Phys. D* **30**, 284–296.
15. Foias, C., Holm, D. D., and Titi, E. S. (2002). The three-dimensional viscous Camassa–Holm equations and their relation to the Navier–Stokes equations and turbulence theory. *J. Dyn. Diff. Eqns.* **14**, 1–34.
16. Foias, C., Manley, O., Rosa, R., and Temam, R. (2001). *Navier–Stokes Equations and Turbulence*. Cambridge University Press.
17. Foias, C., Sell, G. R., and Temam, R. (1985). Variétés inertielles des équations differentielles dissipatives. *C.R. Acad. Sci. Paris, Sér I.* **301**, 139–142.
18. Foias, C., Sell, G. R., and Temam, R. (1988). Inertial manifolds, for nonlinear evolutionary equations. *J. Diff. Eqns.* **73**, 309–353.
19. Holm, D. D., Marsden, J. E., and Ratiu, T. S. (1998). Euler-Poincaré equations and semidirect products with applications to continuum theories. *Adv. Math.* **137**, 1–81.
20. Holm, D. D., Marsden, J. E., and Ratiu, T. S. (1998). Euler-Poincaré models of ideal fluids with nonlinear dispersion. *Phys. Rev. Lett.* **80**, 4173–4176.

21. Karamanos, G. S., and Karniadakis, G. E. (2000). A spectral vanishing viscosity method for large-eddy simulations. *J. Comp. Phys.* **163**, 22–50.
22. Kolmogorov, A. N. (1941). The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers. *C.R.* (*Doklady*) *Acad. Sci. URSS*. **30**, 301–305.
23. Kraichnan, R. H. (1967). Inertial ranges in two-dimensional turbulence. *Phys. Fluids*. **10**, 1417–1423.
24. Kraichnan, R. H. (1976). Eddy viscosity in two and three dimensions. *J. Atmos. Sci.*, **33**, 1521–1536.
25. Kukavica, I. (1995). Level sets of the vorticity and the stream function for the 2D periodic Navier–Stokes equations with potential forces. *J. Diff. Eq.* **126**, 374–388.
26. Kukavica, I. (2003). Length of vorticity nodal sets for solutions of the 2D Navier–Stokes equations. *Comm. PDE* **28**, 771–793.
27. Landau, L., and Lifschitz, E. (1953). *Fluid Mechanics*. Addison-Wesley, New York.
28. Marsden, J. E., and Shkoller, S. (2002). The anisotropic Lagrangian-averaged Euler and Navier–Stokes equations. *Arch. Rat. Mech. Anal.* **166**, 27–46.
29. Sell, G. R. (1996). Global attractors for the three-dimensional Navier–Stokes equations. *J. Dynam. Diff. Eqns.* **8**, 1–33.
30. Tadmor, E. (1989). Convergence of spectral methods for nonlinear conservation laws. *SIAM J. Nimer. Anal.* **26**, 30.
31. Temam, R. (1985). Attractors for Navier–Stokes equations. *Nonlinear Partial Differential Equations and Their Applications*. In Brézis, H., and Lions, J. L. (eds.), *Séminaire du Collège de France*, Vol. 7, Pittman, Boston, pp. 272–292.
32. Temam, R. (1986). Infinite-dimensional dynamical systems in fluid mechanics. In Browder, F. (ed.), *Nonlinear Functional Analysis and its Applications*. Proc. Sympos. Pure Math., Vol. 45, Amer. Math. Soc., Providence, RI, pp. 40–45.
33. Temam, R. (1996). *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*. Springer-Verlag, New York.